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SYMPOSIA IN APPLIED MATHEMATICS
VOLUME III

ELASTICITY

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EDITORIAL COMMITTEE

R. V. Churchill
Eric Reissner
A. H. Taub

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CONTENTS

Editor's Preface v

Approximate methods of Solution of Two-dimensional Problems in Anisotropic Elasticity 1
By I. S. Sokolnikoff

Beams Under Concentrated Loading 13
By G. E. Hay

On Axisymmetrical Deformations of Thin Shells of Revolution 27
By Eric Reissner

On Asymptotic Integration in Shell Theory 53
By F. B. Hildebrand

Some Recent Applications of the Theory of Finite Elastic Deformation 67
By B. R. Seth

Elastic Stability of the Facings of Sandwich Columns 85
By H. W. March

Dynamic Loads on Thin Plates on Elastic Foundations 107
By D. L. Holl

Kirchhoff's Boundary Conditions and the Edge Effect for Elastic Plates 117
By K. O. Friedrichs

Some Problems in the Bending of Thin Plates 125
By G. F. Carrier and F. S. Shaw

Prestressing a Plane Circular Plate to Stiffen It against Buckling (Abstract) 129
By J. J. Stoker

The Bending of Thin Elliptic Plates 131
By C. L. Perry

New Methods for the Estimation of Torsional Rigidity 141
By Alexander Weinstein

Stress Fields of Axially Symmetric Shafts in Torsion and Related Fields 163
By H. Poritsky

Stress Analysis in Elastic-plastic Structures 187
By W. Prager and P. S. Symonds

Elastic-plastic Straining (Abstract) 199
By Sir Richard Southwell

A Graphical Method for Solving Problems in Plane Plasticity 201
By N. Coburn

On Stress Discontinuities in Plane Plastic Flow 213
By E. H. Lee

References to Papers by L. H. Donnell and C. C. Wan, T. J. Jaramillo, C. C. Miesse, and S. Dokos 229

Index 231

EDITOR'S PREFACE

The papers that were presented during the Third Symposium in Applied Mathematics of the American Mathematical Society are published in this volume. The Symposium was held at the University of Michigan from June 14 to 16, 1949. The subject of the Symposium was *Elasticity;* it included plasticity. The Symposium was cosponsored by the Applied Mechanics Division of the American Society of Mechanical Engineers, and joint sessions of the Symposium and that Division's Fifteenth Applied Mechanics Conference were held on June 15.

The papers in this volume are grouped roughly according to subjects. Owing to previous arrangements for publication, two of the papers appear here as abstracts containing references to the complete papers. Four papers from the program of the Applied Mechanics Conference were presented at the joint sessions with the Symposium. The place of publication of those four papers is announced at the end of this volume.

On behalf of the American Mathematical Society, the Editorial Committee, consisting of Eric Reissner, A. H. Taub, and the undersigned, wishes to express its gratitude to the McGraw-Hill Book Company, Inc., for undertaking the publication of this volume. The Committee extends its thanks to R. C. Gibbs of the National Research Council for his assistance in initiating these arrangements for publication and to the officers of McGraw-Hill and of the Society for carrying out this arrangement. In the editorial processing of this volume, the undersigned wishes to acknowledge the work of the New York Office of the Society as well as the assistance of the other two members of the Editorial Committee and the help given by G. E. Hay and P. G. Hodge, Jr.

R. V. CHURCHILL
Chairman, Editorial Committee
Proceedings of Symposia in Applied Mathematics
American Mathematical Society

APPROXIMATE METHODS OF SOLUTION OF TWO-DIMENSIONAL PROBLEMS IN ANISOTROPIC ELASTICITY

BY

I. S. SOKOLNIKOFF

1. The scope of research. This paper contains an account of one phase of research in the domain of two-dimensional problems of anisotropic elasticity initiated at the University of California (Los Angeles) this year. The object of this research is to develop practical methods for the explicit solution of a wide class of problems involving the states of plane stress and plane strain in anisotropic elastic media, and to study the problems of deflection of thin anisotropic elastic plates. The investigation hinges on an approximate determination of appropriate stress functions for the anisotropic problems from certain known stress functions in the corresponding isotropic cases.

A systematic use of Airy's stress function in the solution of two-dimensional problems of anisotropic elasticity, following the pattern laid down by N. I. Muschelisvili for the isotropic case, was made by S. G. Lechnitzky. An outline of this mode of attack on the two-dimensional boundary-value problems of elasticity is contained in an address given by the author [1] before the meeting of the American Mathematical Society in 1941. Since numerous references to original sources are given in that address, they will not be duplicated here, but to facilitate the understanding of the perturbation procedures discussed in Sec. 3, the essential concepts, in so far as they bear on the boundary-value problems of anisotropic elasticity, are sketched in the following section.

2. Formulation of boundary-value problems. We consider a two-dimensional anisotropic elastic medium, having at least one plane of elastic symmetry, which we take as the XY plane of our rectangular coordinate system. The region R occupied by the medium is, in general, multiply connected. We denote the boundary of the region by C, where C may consist of the exterior contour C_0 and several interior contours C_i $(i = 1, \cdots, n)$. The region R (if multiply connected) may be regarded either as representing a thin plate with holes or the cross section of an infinitely long cylinder with longitudinal cavities.

We suppose that either the distribution of stresses along C (the first boundary-value problem) or the displacement of points of the boundary C (the second boundary-value problem), is known. The problem is to determine stresses and strains in the region.

It is well known[1] that the solution of these boundary-value problems hinges on the determination of Airy's stress function $U(x, y)$, satisfying the differential equation of the form

---

[1] See Bibliography on pp. 553–555 of the address cited in [1].

$$c_1 \frac{\partial^4 U}{\partial x^4} + c_2 \frac{\partial^4 U}{\partial x^3 \, \partial y} + c_3 \frac{\partial^4 U}{\partial x^2 \, \partial y^2} + c_4 \frac{\partial^4 U}{\partial x \, \partial y^3} + c_5 \frac{\partial^4 U}{\partial y^4} = 0, \tag{1}$$

where the constants c_i are real functions of the known elastic moduli of the medium.

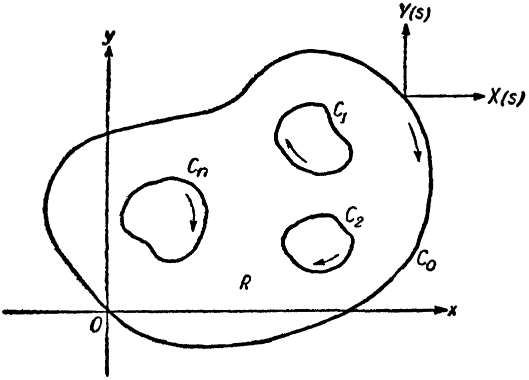

FIG. 1

The components τ_{xx}, τ_{xy}, τ_{yy} of the stress tensor $\boldsymbol{\tau}$ are related to U by the formulas

$$\tau_{xx} = \frac{\partial^2 U}{\partial y^2}, \qquad \tau_{xy} = -\frac{\partial^2 U}{\partial x \, \partial y}, \qquad \tau_{yy} = \frac{\partial^2 U}{\partial x^2}. \tag{2}$$

The general solution of equation (1) has the form

$$U = \sum_{i=1}^{4} F_i(x + \mu_i y), \tag{3}$$

where the functions F_i are of class C^4 and the μ's are the *distinct roots* of the characteristic equation

$$c_5\mu^4 + c_4\mu^3 + c_3\mu^2 + c_2\mu + c_1 = 0. \tag{4}$$

The case of multiple roots of the characteristic equation is of relatively trivial interest because in that case the problem can be reduced to an isotropic one by a simple linear transformation.[2]

From the fact that the energy of deformation is nonnegative, it is easy to show that the roots μ_i of equation (4) are complex numbers. This was first demonstrated by S. G. Lechnitzky.[3]

---

[2] This follows from remarks made immediately after equation (5) below.

[3] See [2], a paper which is concerned with the problem of deflection of thin anisotropic plates.

Since the c_i in equation (4) are real, it follows that its roots are conjugate complex numbers, and we can write

$$\mu_1 = \bar{\mu}_3, \qquad \mu_2 = \bar{\mu}_4.$$

We can rewrite the general solution (3) in a more convenient form by introducing two complex variables z_1 and z_2 defined as follows:

$$\begin{aligned} z_1 &= x + \mu_1 y \\ &\equiv x + (\alpha_1 + i\beta_1)y \\ &\equiv x_1 + iy_1, \end{aligned}$$

where

$$x_1 = x + \alpha_1 y, \qquad y_1 = \beta_1 y;$$

$$\begin{aligned} z_2 &= x + \mu_2 y \\ &\equiv x + (\alpha_2 + i\beta_2)y \\ &\equiv x_2 + iy_2, \end{aligned}$$

with

$$x_2 = x + \alpha_2 y, \qquad y_2 = \beta_2 y.$$

Since $U(x, y)$ is a real function, the solution (3) assumes the form

$$U = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)}, \tag{5}$$

where we use bars to denote the conjugate complex values.

If the characteristic equation (4) has multiple roots, then $z_1 = z_2$, and the solution of equation (1) has the form

$$U = F_1(z_1) + \bar{z}_1 F_2(z_1) + \overline{F_1(z_1)} + \overline{z_1 F_2(z_1)},$$

which becomes identical with the solution of the biharmonic equation when the variable z_1 is replaced by $z = x + iy$. Thus the study of the behavior of anisotropic media, whose elastic properties are such that the roots of the associated characteristic equation are multiple, is reducible to an isotropic case.

It follows from the definition of the variables z_1 and z_2 that their domains R_1 and R_2 are obtainable from R by the homogeneous deformations:

$$T_1: \begin{cases} x_1 = x + \alpha_1 y, \\ y_1 = \beta_1 y, \end{cases}$$

and

$$T_2: \begin{cases} x_2 = x + \alpha_2 y, \\ y_2 = \beta_2 y. \end{cases}$$

From the differentiability of Airy's stress function $U(x, y)$, it follows that $F_1(z_1)$ and $F_2(z_2)$ are analytic functions defined in the regions R_1 and R_2, respectively.

We readily deduce from the representation (5) and relations (2) that

$$\begin{aligned}\tau_{xx} &= 2\mathcal{R}[\mu_1^2F_1''(z_1) + \mu_2^2F_2''(z_2)],\\ \tau_{yy} &= 2\mathcal{R}[F_1''(z_1) + F_2''(z_2)],\\ \tau_{xy} &= -\,2\mathcal{R}[\mu_1F_1''(z_1) + \mu_2F_2''(z_2)],\end{aligned}$$

where $\mathcal{R}$ denotes the real part of expressions contained in the brackets.

If we insert these expressions in the boundary conditions:

$$\begin{aligned}\tau_{xx}\cos(x, n) + \tau_{xy}\cos(y, n) &= X(s),\\ \tau_{xy}\cos(x, n) + \tau_{yy}\cos(y, n) &= Y(s), \qquad \text{on } C,\end{aligned}$$

in which $X(s)$ and $Y(s)$ are the components of prescribed external forces acting on C (s being the arc parameter along C) and integrate along the contours from some initial value $s = s_0$ to a variable point $s = s$, we get[4]

$$\begin{aligned}2\mathcal{R}[\mu_1F_1'(z_1) + \mu_2F_2'(z_2)] &= \int_{s_0}^{s} X(s)\,ds + c_1\\ &\equiv f_1(s),\\ 2\mathcal{R}[F_1'(z_1) + F_2'(z_2)] &= -\int_{s_0}^{s} Y(s)\,ds + c_2\\ &\equiv f_2(s),\end{aligned} \tag{6}$$

where f_1 and f_2 are known functions along C, and c_1 and c_2 are constants. These can be fixed arbitrarily on only one of the contours, say C_0.

If, on the other hand, the components of the displacement vector are specified along C as functions of the arc-parameter s, so that

$$\begin{aligned}u &= g_1(s),\\ v &= g_2(s),\end{aligned}$$

then a somewhat less obvious computation utilizing the stress-strain relations yields [3]

$$\begin{aligned}2\mathcal{R}[a_1F_1'(z_1) + a_2F_2'(z_2)] &= g_1(s),\\ 2\mathcal{R}[b_1F_1'(z_1) + b_2F_2'(z_2)] &= g_2(s), \qquad \text{on } C,\end{aligned} \tag{7}$$

where the a_i and b_i are known rational functions of the elastic moduli of the medium.

It is clear from these formulas that the solution of the first and second boundary-value problems of anisotropic elasticity is reduced to the determination of analytic functions $F_1'(z_1)$ and $F_2'(z_2)$ from functional equations (6) and (7). The questions of existence and uniqueness of solution of these equations are settled by reducing them to certain equivalent systems of integral equations of the standard types.[5] The integral equations can serve as a point of departure for the actual determination of unknown functions, but there are also methods of solution of equations (6) and (7) that are based on an extension of the scheme of N. I. Muschelisvili [4] which proved so successful in the treatment of numerous boundary-value problems of isotropic elasticity. However, in the application of these methods to problems in anisotropic elasticity one frequently runs into serious computational difficulties connected with the construction of suitable conformal mapping functions.[6] It is natural therefore to attempt to reduce the solution of such problems to the isotropic case. This can be done, for certain types of anisotropic media, in the manner outlined in Sec. 3.

---

[4] Note that if n denotes the exterior unit normal, then $dx/ds = -\cos(y, n)$ and $dy/ds = \cos(x, n)$.

The results summarized in this section have been obtained during the past decade or so by several Russian investigators, notably S. G. Lechnitzky. However, many Soviet publications in which these researches appeared were not made accessible to workers in the English-speaking countries. As a consequence, results obtained by the Soviet investigators have been extensively duplicated in this country and in England.[7]

3. Perturbation methods. The fundamental idea underlying the reduction of solution of the boundary-value problems of anisotropic elasticity to the solution of a sequence of isotropic problems is simple. It is a variant of the perturbation procedure often used to obtain approximate solutions of nonlinear differential equations.

In order to simplify the exposition, we consider, instead of equation (1), the equation

$$c_1 \frac{\partial^4 U}{\partial x^4} + c_3 \frac{\partial^4 U}{\partial x^2\, \partial y^2} + c_5 \frac{\partial^4 U}{\partial y^4} = 0, \tag{8}$$

which corresponds to the case of orthotropic elastic medium.[8] The corresponding treatment for the more general case typified by equation (1) presents no complication and merely leads to the more involved recursion relations.

We observe that, if the medium is isotropic, equation (8) reduces to the biharmonic equation

$$\nabla^4 U = 0,$$

---

[5] These matters have been dealt with by S. G. Michlin, Publications de l'institut séismologique, Acad. Sci. URSS, No. 76 (1936). D. I. Sherman, ibid., No. 86 (1938).

[6] See, for example, [5].

[7] See, in particular, numerous papers by A. E. Green, S. Holgate, A. C. Stevenson, and G. I. Taylor in the Royal Proceedings for the year 1945.

[8] It may help to recall that if equation (8) refers to a problem dealing with the state of plane stress, the constants are related to the elastic moduli as follows: $c_1 = 1/E_y$, $c_3 = 1/\mu_{xy} - (2\sigma_{xy})/E_x$, $c_5 = 1/E_x$, where E_x and E_y are Young's moduli in the principal directions indicated by the subscripts, σ_{xy} is Poisson's ratio representing the contraction in the Y direction by a tensile stress acting in the X direction, μ_{xy} is the shearing modulus characterizing the change in angle between the principal directions, and $E_x\sigma_{yx} = E_y\sigma_{xy}$. Similar relations, involving elastic moduli associated with the Z direction, can be written for the c_i's in problems concerned with the state of plane strain.

in which case the roots of characteristic equation (4) are $\mu_1 = \mu_2 = i$. We also note that the μ_i characterize the nature of anisotropy of the medium inasmuch as the coefficients c_i are symmetric functions of the roots μ_i. Thus, if we define the numbers ϵ_i (in general complex) by the formulas

$$\begin{aligned} \mu_1 &= i(1 + \epsilon_1), & \mu_3 &= \bar{\mu}_1, \\ \mu_2 &= i(1 + \epsilon_2), & \mu_4 &= \bar{\mu}_2, \end{aligned}$$

we see that the parameters ϵ_i can be taken as the measure of deviation of the medium from isotropic state. For many structural materials, the deviation parameters ϵ_i are less than 1 in absolute value, and this suggests that the solution of equation (8) might be represented in the form

$$U = \sum_{i,j=0}^{\infty} U_{ij}(x, y)\epsilon_1^i\epsilon_2^j, \tag{9}$$

where the U_{ij} are unknown functions of x and y to be determined so that U satisfies equation (8). If $| \epsilon_i | \ll 1$, a few terms of the series (9) might be expected to yield a good approximation to the desired solution.

Replacing the coefficients c_i in equation (8) by their values in terms of the ϵ_i gives

$$(1 + \epsilon_1)^2(1 + \epsilon_2)^2 \frac{\partial^4 U}{\partial x^4} + [(1 + \epsilon_1)^2 + (1 + \epsilon_2)^2] \frac{\partial^4 U}{\partial x^2\, \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0,$$

and the substitution of the series (9) in this equation leads to the following system of equations to be satisfied by the functions U_{ij}:

$$\begin{aligned} &\nabla^4 U_{00} = 0, \\ &\nabla^4 U_{ij} = -\frac{\partial^2}{\partial x^2} \nabla^2(2U_{i-1\,j} + 2U_{i\,j-1} + U_{i-2\,j} + U_{i\,j-2}) \\ &\qquad\qquad - \frac{\partial^4}{\partial x^4} (2U_{i-1\,j-2} + 2U_{i-2\,j-1} + 4U_{i-1\,j-1} + U_{i-2\,j-2}), \\ &U_{ij} = 0 \qquad\qquad \text{(if } i \text{ or } j \text{ is negative).} \end{aligned} \tag{10}$$

We see that these equations are of the type

$$\nabla^4 U_{ij} = f_{ij}(U_{00}, U_{01}, \cdots, U_{i-1\,j-1}) \qquad (i, j = 1, 2, \cdots), \tag{11}$$

in which the f_{ij} are known functions of the indicated arguments and $U_{ij} = U_{ji}$.

The general solution of the first equation in the system (10) has the form

$$U_{00} = \mathcal{R}[\varphi_{000}(z) + \bar{z}\varphi_{001}(z)],$$

where φ_{000} and φ_{001} are arbitrary analytic functions of $z = x + iy$. The substitution of this expression for U_{00} in the right-hand member of the second equation of the system (10) with $i = 1, j = 0$ yields the equation

$$\nabla^4 U_{10} = -2 \frac{\partial^2}{\partial x^2} \nabla^2 U_{00},$$

whose solution is

$$U_{10} = \mathcal{R}[\varphi_{100}(z) + \bar{z}\varphi_{101}(z) - \tfrac{1}{4}\bar{z}^2\varphi'_{001}(z)],$$

in which the functions φ are analytic. Setting $i = 2, j = 0$ in equation (10) gives the equation for U_{20}, which can be solved by quadratures to yield

$$U_{20} = \mathcal{R}[\varphi_{200}(z) + \bar{z}\varphi_{201}(z) + \tfrac{1}{8}\bar{z}^2\varphi'_{001}(z) - \tfrac{1}{4}\bar{z}^2\varphi'_{101}(z) + \tfrac{1}{24}\bar{z}^3\varphi''_{001}(z)].$$

Continuation of this procedure leads to the general solution of the system (10) in the form

$$U_{ij} = \mathcal{R}\sum_{m=0}^{i+j+1} \bar{z}^m\varphi_{ijm}(z), \tag{12}$$

in which the functions φ_{ij0} and φ_{ij1} are arbitrary and the remaining ones satisfy the relations:

$$\begin{aligned}
&-16(m+2)(m+1)\varphi''_{i\,j\,m+2}\\
&\quad= (m+1)(8\varphi'''_{i-1\,j\,m+1} + 8\varphi'''_{i\,j-1\,m+1} + 4\varphi'''_{i-2\,j\,m+1} + 4\varphi'''_{i\,j-2\,m+1})\\
&\quad+ (m+2)(m+1)(16\varphi''_{i-1\,j\,m+2} + 16\varphi''_{i\,j-1\,m+2} + 8\varphi''_{i-2\,j\,m+2} + 8\varphi''_{i\,j-1\,m+2})\\
&\quad+ (m+3)(m+2)(m+1)(8\varphi'_{i-1\,j\,m+3} + 8\varphi'_{i\,j-1\,m+3} + 4\varphi'_{i-2\,j\,m+3} + 4\varphi'_{i\,j-1\,m+3})\\
&\quad+ 2(\varphi^{(iv)}_{i-1\,j-2\,m} + 2\varphi^{(iv)}_{i-2\,j-1\,m+4} + \varphi^{(iv)}_{i-1\,j-1\,m} + \varphi^{(iv)}_{i-2\,j-2\,m})\\
&\quad+ (m+1)(8\varphi'''_{i-1\,j-2\,m+1} + 8\varphi'''_{i-2\,j-1\,m+1} + 16\varphi'''_{i-1\,j-1\,m+1} + 4\varphi'''_{i-2\,j-2\,m+1})\\
&\quad+ (m+2)(m+1)(12\varphi''_{i-1\,j-2\,m+2} + 12\varphi''_{i-2\,j-1\,m+2} + 24\varphi''_{i-1\,j-1\,m+2}\\
&\quad+ 6\varphi''_{i-2\,j-2\,m+2}) + (m+3)(m+2)(m+1)\\
&\qquad\qquad\cdot(8\varphi'_{i-1\,j-2\,m+3} + 8\varphi'_{i-2\,j-1\,m+3} + 16\varphi'_{i-1\,j-1\,m+3} + 4\varphi'_{i-2\,j-2\,m+3})\\
&\quad+ (m+4)(m+3)(m+2)(m+1)\\
&\qquad\qquad\cdot(2\varphi_{i-1\,j-2\,m+4} + 2\varphi_{i-2\,j-1\,m+4} + 4\varphi_{i-1\,j-1\,m+4} + \varphi_{i-2\,j-2\,m+4}).
\end{aligned}$$

The functions φ_{ij0} and φ_{ij1} are determined (essentially uniquely) by the boundary conditions. We do not write out the corresponding system of functional equations for their determination,[9] because its structure is entirely similar to equations (6) and (7).

Despite the formidable appearance of solution as given by the expression (9), specific computations indicate that a satisfactory approximate solution is obtained, in a number of problems, if one retains in the expansion only linear or quadratic terms in the deviation parameters ϵ_i. The computational labor in such cases is not excessively heavy.

The perturbational procedure, described above, naturally is not the only one that can be applied to anisotropic problems. Thus, one can introduce the perturbation parameters ϵ_i not in the roots μ_i of characteristic equations, but directly into the coefficients of equation (8) after it has been divided through by the numerically largest coefficient c_i. For example, if equation (8) is written in the form

$$2\frac{c_1}{c_3}\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2\,\partial y^2} + 2\frac{c_5}{c_3}\frac{\partial^4 U}{\partial y^4} = 0,$$

---

[9] **A discussion of these equations is too lengthy to be included in this paper.**

and if we set

$$1 - \epsilon_x \equiv 2\frac{c_1}{c_3},$$

$$1 - \epsilon_y \equiv 2\frac{c_5}{c_3},$$

the equation assumes the form

$$(13) \qquad (1 - \epsilon_x)\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2\,\partial y^2} + (1 - \epsilon_y)\frac{\partial^4 U}{\partial y^4} = 0,$$

where the ϵ_x and ϵ_y are new deviation parameters.[10]

If ϵ_x and ϵ_y are small compared with unity, it is reasonable to seek solution of equation (13) in the form

$$(14) \qquad U(x,y) = \sum_{i,j=0}^{\infty} U_{ij}(x,y)\epsilon_x^i\epsilon_y^j.$$

This time the procedure that has led to equations (10) leads to a simpler system of equations:

$$(15) \qquad \begin{aligned} &\nabla^4 U_{00} = 0, \\ &\nabla^4 U_{ij} = \frac{\partial^4}{\partial x^4}U_{i-1\,j} + \frac{\partial^4}{\partial y^4}U_{i\,j-1} \qquad (i, j = 1, 2, \cdots). \end{aligned}$$

Since the formal structure of these equations is typified by the system (11), we again seek the solution in the form (12) and obtain the following relations for the functions $\varphi_{ij\;m+2}$ $(m = 0, 1, 2, \cdots)$:

$$\begin{aligned} 16(m+2)!\varphi''_{i\,j\,m+2} &\\ = [m!\varphi^{(iv)}_{i-1\,j\,m} &+ 4(m+1)!\varphi'''_{i-1\,j\,m+1} + 6(m+2)!\varphi''_{i-1\,j\,m+2} \\ &+ 4(m+3)!\varphi'_{i-1\,j\,m+3} + (m+4)!\varphi_{i-1\,j\,m+4}] \\ + [m!\varphi^{(iv)}_{i\,j-1\,m} &- 4(m+1)!\varphi'''_{i\,j-1\,m+1} + 6(m+2)!\varphi''_{i\,j-1\,m+2} \\ &- 4(m+3)!\varphi'_{i\,j-1\,m+3} + (m+4)!\varphi_{i\,j-1\,m+4}]. \end{aligned}$$

As in the former case, the functions φ_{ij0} and φ_{ij1} are subject to the determination from prescribed boundary conditions.[11]

Another variant of the perturbation scheme, leading to simple recursion formulas, makes use of equation (8) in the form[12]

$$(16) \qquad \frac{\partial^4 U}{\partial x^4} + 2\kappa\frac{\partial^4 U}{\partial x\,\partial \eta} + \frac{\partial^4 U}{\partial \eta^4} = 0,$$

---

[10] These parameters are obviously functions of ϵ_1 and ϵ_2 used in the preceding discussion, since the coefficients in equation (13) are symmetric functions of the roots μ_i of the characteristic equation.

[11] These computations were performed by Mr. Harold Luxenberg, Research Assistant, University of California (Los Angeles) who made use of them in the analysis of deflection of thin orthotropic elastic plates. In particular, this variant of the perturbational procedure leads readily to the known solution of the problem of deflection of a clamped elliptic plate subjected to a uniform normal load.

[12] This variant was considered by Mr. Julius Brandstatter, Research Assistant, University of California (Los Angeles), who is responsible for the calculations given in the remainder of this section.

which is obtained from equation (8) by making the transformation

$$\eta = \left(\frac{c_1}{c_5}\right)^{1/4} y.$$

The value of κ in terms of the original coefficients c_i is given by the expression $\kappa = c_3/(c_1c_5)^{1/2}$.

There are two cases to be considered, depending on whether $|\kappa|$ is less than or greater than 1.

If $|\kappa| < |$, we set $\kappa = 1 - \epsilon$ and seek the solution in the form

$$U = \sum_{i=0}^{\infty} U_i(x, \eta)\epsilon^i. \tag{17}$$

The substitution of this assumed solution in equation (16) leads to the following relations to be satisfied by the functions $U_i(x, \eta)$:

$$\nabla^4 U_i = 2\frac{\partial^4 U_{i-1}}{\partial x^2\, \partial \eta^2}, \tag{18}$$

where

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \eta^2}\right).$$

Now, if we introduce a complex variable $z = x + i\eta$, we can rewrite the relations (18) in the form

$$\nabla^4 U_i = -2\left(\frac{\partial^4}{\partial z} - 2\,\frac{\partial^4}{\partial z\,\partial \bar z} + \frac{\partial^4}{\partial \bar z^4}\right) U_{i-1},$$

which are satisfied by

$$U_i = \Re \sum_{m=0}^{2i+1} \varphi_{im}(z)\bar z, \tag{19}$$

in which the $\varphi_{im}(z)$ are analytic functions of z.

It is easy to verify that the functions φ_{im} must satisfy the conditions

$$\begin{aligned} 8(m+2)(m+1)\varphi_{i\ m+2}'' = {} & -\varphi_{i-1\ m}^{(iv)} + 2(m+2)(m+1)\varphi_{i-1\ m+2}'' \\ & - (m+4)(m+3)(m+2)(m+1)\varphi_{i-1\ m+4}. \end{aligned}$$

If $|\kappa| > 1$, we set $\kappa = 1/(1 - \epsilon)$ and again seek the solution in the form (17). This time the U_i satisfy the differential equation

$$\nabla^4 U_i = \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial \eta}\right) U_{i-1},$$

or

$$\nabla^4 U_i = 2\left(\frac{\partial^4}{\partial z^4} + 6\,\frac{\partial^4}{\partial z^2\,\partial \bar z^2} + \frac{\partial^4}{\partial \bar z^2}\right) U_{i-1},$$

where $z = x + i\eta$ and the functions $\varphi_{im}(z)$ in (19) are bound by the relations:

$$\begin{aligned} 8(m+2)(m+1)\varphi_{i\ m+2}'' = {} & \varphi_{i-1\ m}^{(iv)} + 6(m+2)(m+1)\varphi_{i-1\ m+2}'' \\ & + (m+4)(m+3)(m+2)(m+1)\varphi_{i-1\ m+4}. \end{aligned}$$

In both of the foregoing cases φ_{i0} and φ_{i1} are arbitrary functions determined by the boundary conditions.

We have already remarked that the first of the perturbation procedures described in this section can be applied to the general case of anisotropic media with only one plane of elastic symmetry which is typified by equation (1).

A special case of this procedure, applied to equation (16) is of interest.

The characteristic equation (4) associated with equation (16) is

$$\mu^4 + 2\kappa\mu^2 + 1 = 0.$$

If $\kappa > 1$, the roots of this equation are pure imaginary, and we consider the case for which the deviation parameters ϵ_i defined by

$$\mu_1 = i(1 + \epsilon_1), \qquad \mu_2 = i(1 + \epsilon_2), \qquad \mu_3 = \bar{\mu}_1, \qquad \mu_4 = \bar{\mu}_2$$

are less than unity. The ϵ_i are related to κ by the formula

$$2\kappa = 2(1 + \epsilon_1 + \epsilon_2) + \epsilon_1^2 + \epsilon_2^2.$$

We assume that U is represented in the double series

$$U = \sum_{i,j=0}^{\infty} U_{ij}(x, \eta)\epsilon_1^i\epsilon_2^j,$$

with $U_{ij} = U_{ji}$, and find that

$$\nabla^4 U_{ij} = -\frac{\partial^4}{\partial x^2\,\partial\eta^2}\,(2U_{i-1\,j} + 2U_{i\,j-1}U_{i-2\,j} + U_{i\,j-2}),$$

or

$$\nabla^4 U_{ij} = \left(\frac{\partial^4}{\partial z^4} - 2\,\frac{\partial^4}{\partial z^2\,\partial\bar{z}^2} + \frac{\partial^4}{\partial\bar{z}^4}\right)(2U_{i-1\,j} + 2U_{i\,j-1} + U_{i-2\,j} + U_{i\,j-2}),$$

where $z = x + i\eta$.

These equations have solutions of the form

$$U_{ij} = \mathcal{R}\sum_{m=0}^{2i+2j+1} \varphi_{ijm}(z)\bar{z}^m,$$

where the analytic functions φ_{ijm} satisfy the recursion relations:

$$\begin{aligned}
16(m + 2)(m + 1)&\varphi''_{i\,j\,m+2}\\
&= 2\varphi^{(iv)}_{i-1\,j\,m} + 2\varphi^{(iv)}_{i\,j-1\,m} + \varphi^{(iv)}_{i-2\,j\,m} + \varphi^{(iv)}_{i\,j-2\,m}\\
&\quad - 2(m + 2)(m + 1)(2\varphi''_{i-1\,j\,m+2} + 2\varphi''_{i\,j-1\,m+2} + \varphi''_{i-2\,j\,m+2} + \varphi''_{i\,j-2\,m+1})\\
&\quad + (m + 4)(m + 3)(m + 2)(m + 1)(2\varphi_{i-1\,j\,m+4} + 2\varphi_{i\,j-1\,m+4} + \varphi_{i-2\,j\,m+4}\\
&\qquad + \varphi_{i\,j-2\,m+4}).
\end{aligned}$$

The functions φ_{ij0} and φ_{ij1} remain arbitrary and are subject to the determination from the boundary conditions.

It should be emphasized that the perturbational procedures described in this section cannot be expected to yield satisfactory solutions where the media under consideration are highly anisotropic. However, computations now in progress indicate that useful approximate solutions can be obtained even for such anisotropic materials as wood.

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BEAMS UNDER CONCENTRATED LOADING

BY

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1. Introduction. Let us consider a beam which is elastically isotropic and homogeneous, with a uniform cross section. Body forces are assumed negligible, and the only loads applied to the beam are normal to the generators of the surface of the beam and are distributed uniformly along the beam. The beam is in a state of plane strain.

Rectangular Cartesian coordinates x_1, x_2, x_3 are introduced, as shown in Fig. 1. We adopt the convention that a Latin subscript ranges over the values 1, 2, 3. Accordingly, we can denote the coordinates mentioned above by the symbol x_i. We can also denote the corresponding components of displacement, strain and stress by u_i, e_{ij}, T_{ij}, respectively. The components of displacement then have the form

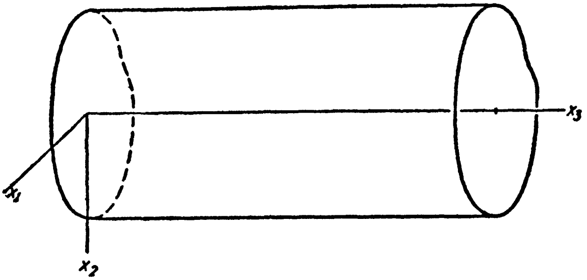

FIG. 1

$$u_1 = u_1(x_1, x_2), \qquad u_2 = u_2(x_1, x_2), \qquad u_3 = 0, \tag{1}$$

and the stress components are given by the relations

$$\begin{gathered} T_{11} = \frac{\partial^2 U}{\partial x_2^2}, \qquad T_{22} = \frac{\partial^2 U}{\partial x_1^2}, \qquad T_{12} = -\frac{\partial^2 U}{\partial x_1 \, \partial x_2}, \\ T_{13} = T_{23} = 0, \qquad T_{33} = \sigma(T_{11} + T_{22}), \end{gathered} \tag{2}$$

where σ is Poisson's ratio, and U is a function of x_1, x_2 satisfying the biharmonic differential equation

$$\nabla^4 U = 0 \tag{3}$$

throughout the cross section of the beam. The operator ∇^4 has the usual meaning, namely,

$$\nabla^4 = \nabla^2\nabla^2, \qquad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

The function U is Airy's stress function. The determination of the stresses in the beam is thus reduced to the determination of a function $U(x_1, x_2)$ which is biharmonic throughout the cross section of the beam and satisfies certain boundary conditions on the edge of the cross section, the precise form of these boundary conditions depending on the manner in which the sides of the beam are loaded or displaced.

There are two approaches to the problem of finding the function U. In one of these, equations (2) and (3) are expressed in terms of curvilinear coordinates which fit the boundaries of the cross sections. This can be done conveniently by the use of tensor calculus. Infinite sets of biharmonic functions are then obtained by separation of variables or by some similar technique, and U is determined finally as an infinite series of these functions. Many problems have been solved in this way in the cases of polar coordinates [1, 2, 3], elliptic coordinates [4], and bipolar coordinates [5, 6].

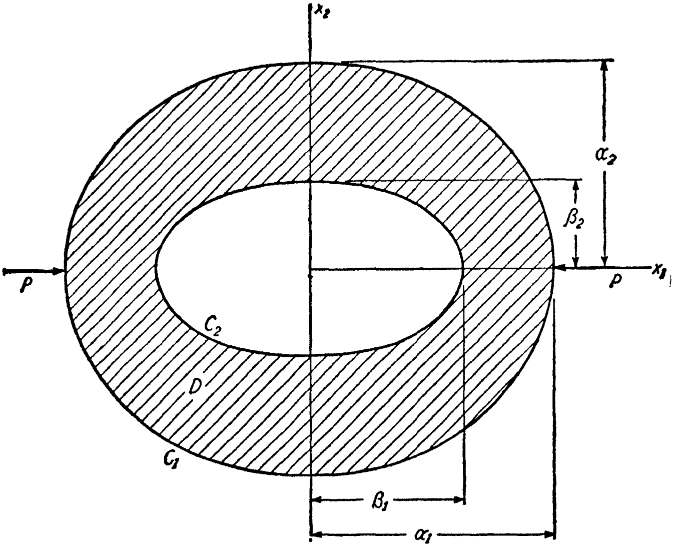

Fig. 2

The second approach to the problem of finding U is the one usually associated with the name N. I. Muschelisvili [7]. Here an extensive use of complex variables and conformal mapping is made. More recently this second approach has been developed independently [8].

2. The problem. In this paper we shall consider the case when the cross section D of the beam is bounded by two confocal ellipses C_1 and C_2 as shown in Fig. 2. Also, we shall consider the case when the loading is normal to the surface of the beam and acts only at the two ends of the major axis of C_1. The load acting on a general cross section then consists of the two equal and opposite concentrated forces shown in Fig. 2. The magnitude of these forces, per unit length of the beam, will be denoted by the constant P.

The corresponding problem in the case when the confocal ellipses degenerate into concentric circles has been considered by a number of investigators [1, 3] by the use of curvilinear coordinates.

3. The method of curvilinear coordinates. Let us introduce the elliptic coordinates ξ_1, ξ_2 by the relation

$$x_1 + ix_2 = c \cosh (\xi_1 + i\xi_2),$$

where c is a real constant and i is the imaginary unit. The parametric lines of ξ_1 and ξ_2 are confocal ellipses and hyperbolas, the rectangular Cartesian coordinates of the foci being $(\pm c, 0)$. Equations (2) and (3) are then transformed into elliptic coordinates; this can be done very easily by the use of tensor calculus.

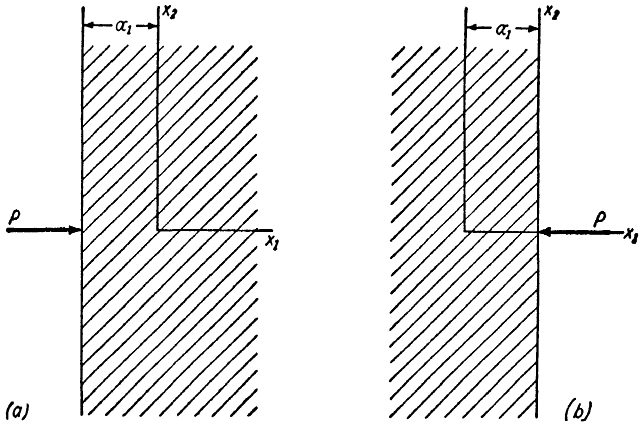

FIG. 3

Standard procedure indicates that the next step is to assume for the stress function U an infinite series of the biharmonic functions in elliptic coordinates deduced by Timpe [4], the coefficients in this series being unknown constants. These unknown coefficients are then deduced so that the boundary conditions are satisfied; this last step entails the expansion of the normal and shear loads in Fourier series in ξ_2. However the Fourier expansion of the load acting in the problem outlined in Sec. 2 is divergent. This difficulty can be avoided by a method due to Timoshenko [1, p. 119]. This method is as follows:

Let the length of the major axis of the outer boundary be $2\alpha_1$. Let S denote the desired system of stresses, and let us write $S = S_1 + S_2 + S_3$ where S_1 and S_2 are the systems of stresses in the semiinfinite solids $x_1 \geqq -\alpha_1$ and $x_1 \leqq \alpha_1$ acted upon by the concentrated loads P as shown in Fig. 3, and S_3 is to be chosen so that the boundary conditions on S are satisfied. The systems S_1 and S_2 are simple radial distributions and can be written down at once [9]. The system S_3 is then sought as an infinite series of Timpe's biharmonic functions in elliptic coordinates. The writer has attempted to determine S_3 in this way, but the attempt was soon abandoned because of severe algebraic complications due chiefly to the fact that S_1 and S_2 cannot be expressed simply in elliptic coordinates.

4. The method of Muschelisvili. A detailed development of this method, with applications, has been given by I. S. Sokolnikoff [7]. In the present section a short description of this method will be given.

Let the complex variable z be defined by the relation

$$z = x_1 + ix_2.$$

Then the general biharmonic function U can be expressed in the form

$$2U = \bar{z}\phi(z) + z\bar{\phi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z}), \tag{4}$$

where $\phi(z)$, $\chi(z)$ are functions of z with real and imaginary components satisfying Cauchy-Riemann equations everywhere in the cross section, and superimposed bars denote complex conjugates. The functions $\phi(z)$, $\chi(z)$ are thus analytic in simply connected cross sections but are not necessarily analytic in multiply connected cross sections. The determination of U is thus reduced to the determination of $\phi(z)$ and $\chi(z)$. The components of stress, strain, and displacement can be expressed explicitly in terms of ϕ and χ. In fact, it can be shown that

$$T_{11} + T_{22} = 2[\phi'(z) + \bar{\phi}'(\bar{z})], \tag{5}$$

$$T_{22} - T_{11} + 2iT_{12} = 2[\bar{z}\phi''(z) + \psi'(z)], \tag{6}$$

where primes denote derivatives, and

$$\psi(z) = \chi'(z). \tag{7}$$

Also, for the displacements we have

$$u_1 + iu_2 = \frac{1+\sigma}{E}[(3 - 4\sigma)\phi(z) - z\bar{\phi}'(\bar{z}) - \bar{\psi}(\bar{z})], \tag{8}$$

where σ is Poisson's ratio and E is Young's modulus.

Let C be any curve in the general cross section, running from a point A to a point B. Let us consider the cylinder with C as directrix and with generators parallel to the x_3 axis. The force system acting across a unit length of this cylinder has a resultant with components F_1, F_2 given by the relation

$$F_1 + iF_2 = -i[\phi(z) + z\bar{\phi}'(\bar{z}) + \bar{\psi}(\bar{z})]_A^B. \tag{9}$$

If the displacements in the beam are specified to within an arbitrary rigid body displacement, then $\phi(z)$ and $\psi(z)$ are determined to within expressions of the form $Ciz + \alpha + i\beta$ and $\alpha' + i\beta'$, respectively, where C, α, β, α', β' are arbitrary real constants. If the displacements in the beam are completely specified, then

$$C = 0, \qquad (3 - 4\sigma)\alpha - \alpha' = 0, \qquad (3 - 4\sigma)\beta + \beta' = 0, \tag{10}$$

so that either $\phi(z)$ or $\psi(z)$ is determined to within an arbitrary complex constant.

5. Forms of $\phi(z)$ and $\psi(z)$. Let us suppose that a single concentrated force with components L_1, L_2 acts at a point z_0 on the boundary of the cross section of a beam, as shown in Fig. 4. We write

$$z_1 = z - z_0 = r_1 e^{i\theta_1}. \tag{11}$$

Let $\phi_1(z_1)$, $\psi_1(z_1)$ be the complex stress functions corresponding to the state of stress. The state of stress near z_0 is simply radial [9], and hence T_{ij} become infinite at z_0 like $1/r_1$. Thus, from (5) and (6) it follows that $\phi_1'(z_1)$ and $\psi_1'(z_1)$ have simple poles at z_0. Hence we can write

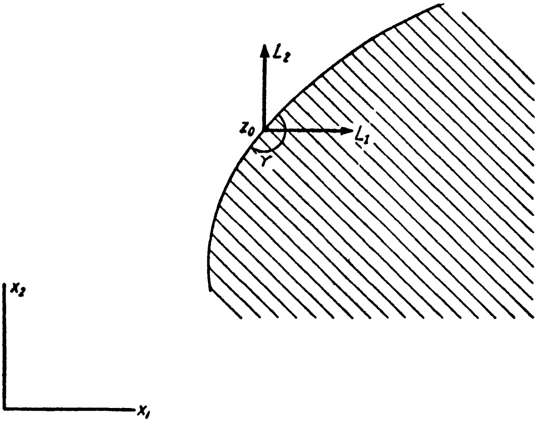

FIG. 4

$$\phi_1'(z_1) = \sum_{n=-1}^{\infty} e_n z_1^n, \qquad \psi_1'(z) = \sum_{n=-1}^{\infty} f_n z_1^n. \tag{12}$$

We now isolate the point z_0 by means of the semicircle γ shown. The resultant of the forces acting across γ can now be found by means of equation (9), in terms of e_n and f_n. If F_1, F_2 denote the components of this resultant, then we must have

$$L_1 + iL_2 = -\lim_{r_1 \to 0} (F_1 + iF_2),$$

whence it is found that

$$e_{-1} = -\frac{1}{2\pi}(L_1 + iL_2), \qquad f_{-1} = \frac{1}{2\pi}(L_1 - iL_2).$$

Thus

$$
\begin{aligned}
\phi_1'(z_1) &= -\frac{1}{2\pi}\frac{L_1 + iL_2}{z_1} + \phi_2(z_1), \\
\psi_1'(z_1) &= \frac{1}{2\pi}\frac{L_1 - iL_2}{z_1} + \psi_2(z_1),
\end{aligned}
\tag{13}
$$

where $\phi_2(z_1)$ and $\psi_2(z_1)$ are analytic everywhere in the cross section.

The functions $\phi_1'(z_1)$ and $\psi_1'(z_1)$ describe the state of stress in terms of a coordinate system with origin at z_0. We now transform to the original coordinate system. Let $\phi(z)$ and $\psi(z)$ describe the state of stress in terms of the original coordinates. Since this transformation leaves the components of stress unaltered, then by (5) and (6) we have

$$
\begin{aligned}
\phi'(z) + \bar{\phi}'(\bar{z}) &= \phi_1'(z_1) + \bar{\phi}_1'(\bar{z}_1), \\
\bar{z}\phi''(z) + \psi'(z) &= \bar{z}_1\phi_1''(z_1) + \psi_1'(z_1).
\end{aligned}
\tag{14}
$$

In Sec. 4 it was indicated that $\phi'(z)$ is determined only to within an imaginary complex constant. Hence (14) yields

$$
\begin{aligned}
\phi'(z) &= \phi_1'(z_1) = \phi_1'(z - z_0), \\
\psi'(z) &= -\bar{z}_0\phi_1''(z - z_0) + \psi_1'(z - z_0),
\end{aligned}
$$

and so we have, by use of (13),

$$
\begin{aligned}
\phi'(z) &= -\frac{1}{2\pi}\frac{L_1 + iL_2}{z - z_0} + \phi_3(z), \\
\psi'(z) &= -\frac{\bar{z}_0}{2\pi}\frac{L_1 + iL_2}{(z - z_0)^2} + \frac{1}{2\pi}\frac{L_1 - iL_2}{z - z_0} + \psi_3(z),
\end{aligned}
\tag{15}
$$

whence

$$
\begin{aligned}
\phi(z) &= -\frac{1}{2\pi}(L_1 + iL_2)\log(z - z_0) + \phi_4(z), \\
\psi(z) &= \frac{\bar{z}_0}{2\pi}\frac{L_1 + iL_2}{z - z_0} + \frac{1}{2\pi}(L_1 + iL_2)\log(z - z_0) + \psi_4(z),
\end{aligned}
\tag{16}
$$

where $\phi_3(z)$, $\psi_3(z)$, $\phi_4(z)$, $\psi_4(z)$ are analytic everywhere in the cross section D.

Let us now return to the hollow beam the cross section of which is shown in Fig. 2. Because of (16), the function $\phi(z)$ describing the stress in this beam has the form

$$
\phi(z) = -\frac{P}{2\pi}\log(z + \alpha_1) + \frac{P}{2\pi}\log(z - \alpha_1) + \phi_0(z),
$$

or

$$
\phi(z) = \frac{P}{2\pi}\log\frac{z - \alpha_1}{z + \alpha_1} + \phi_0(z),
\tag{17}
$$

where $\phi_0(z)$ is analytic everywhere in the cross section D. Similarly, $\psi(z)$ is found to have the form

$$
\psi(z) = -\frac{P\alpha_1}{\pi}\frac{z}{z^2 - \alpha_1^2} - \frac{P}{2\pi}\log\frac{z - \alpha_1}{z + \alpha_1} + \psi_0(z)
\tag{18}
$$

where $\psi_0(z)$ is analytic everywhere in the cross section D.

The functions $\phi_0(z)$ and $\psi_0(z)$ are to be chosen so that the sides of the beam are free from loading. Because of equation (9), we see that this condition is satisfied if

$$\phi(z) + z\bar{\phi}'(\bar{z}) + \bar{\psi}(\bar{z}) = \text{const.} \tag{19}$$

on the ellipses C_1 and C_2 bounding the cross section D.

6. A conformal mapping. In order to utilize the boundary condition (19) conveniently, it seems advisable to map the cross section D of the beam conformally onto the region d between two concentric circles.

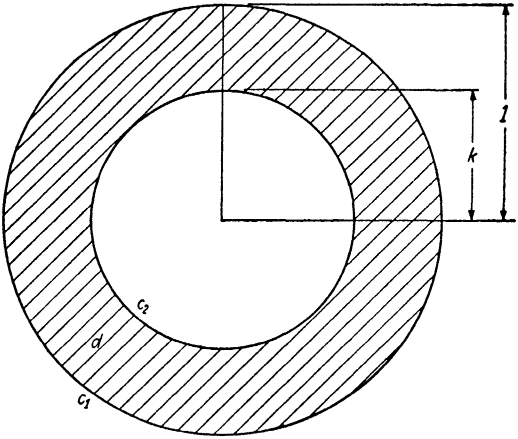

FIG. 5

We introduce the complex variable

$$\zeta = \rho e^{i\theta}, \tag{20}$$

and the mapping relation

$$z = R\left(\zeta + \frac{m}{\zeta}\right) = \omega(\zeta) \qquad (R > 0, 0 < m < 1), \tag{21}$$

where R and m are constants. This mapping relation maps a family of confocal ellipses in the z plane onto a family of concentric circles of radii ρ such that $\rho \geqq \sqrt{m}$. The circle $\rho = \sqrt{m}$ is the map of the degenerate ellipse consisting of the line segment joining the foci. The mapping relation (21) then maps the cross section of the beam onto the region d bounded by circles c_1 and c_2, as shown in Fig. 5. This region is such that $k \leqq \rho \leqq 1$, where k is a positive constant such that

$$k > \sqrt{m}. \tag{22}$$

We must choose R, m, k so that

$$\alpha_1 = R(1+m), \qquad \alpha_2 = R(1-m), \tag{23}$$

$$\beta_1 = R\left(k + \frac{m}{k}\right), \qquad \beta_2 = R\left(k - \frac{m}{k}\right), \tag{24}$$

where α_1, α_2, β_1, β_2 are the semiaxes of the ellipses C_1 and C_2 bounding the cross section of the beam. Since C_1 and C_2 are confocal, only three of α_1, α_2, β_1, β_2 can be assigned at will, and so the four equations (23) and (24) are not independent. They can be solved for a unique set of values of R, m, k satisfying the inequality (22).

Because of equation (21), we can write

$$\phi(z) = \phi[\omega(\zeta)] = \Phi(\zeta), \qquad \psi(z) = \psi[\omega(\zeta)] = \Psi(\zeta). \tag{25}$$

The forms of $\phi(z)$ and $\psi(z)$ are given by (17) and (18). We shall now deduce the forms of $\Phi(\zeta)$ and $\Psi(\zeta)$.

Because of (21) and (23) we can show that

$$\frac{z-\alpha_1}{z+\alpha_1} = \frac{(\zeta-1)(\zeta-m)}{(\zeta+1)(\zeta+m)}.$$

Since the points $\zeta = \pm m$ are inside the circle c_2, the function $\log[(\zeta-m)/(\zeta+m)]$ is analytic everywhere in d and so (17) yields

$$\Phi(\zeta) = \frac{P}{2\pi}\log\frac{\zeta-1}{\zeta+1} + \frac{P}{2\pi}\Phi_0(\zeta), \tag{26}$$

where $\Phi_0(\zeta)$ is analytic everywhere in d. Similarly, we can show that

$$\Psi(\zeta) = -\frac{P}{\pi}\frac{1+m}{1-m}\frac{\zeta}{\zeta^2-1} - \frac{P}{2\pi}\log\frac{\zeta-1}{\zeta+1} + \frac{P}{2\pi}\Psi_0(\zeta), \tag{27}$$

where $\Psi_0(\zeta)$ is analytic everywhere in d.

7. Boundary conditions. The boundary condition which must be satisfied on both boundaries C_1 and C_2 is (19). Since

$$\phi'(z) = \frac{\Phi'(\zeta)}{\omega'(\zeta)},$$

this boundary condition can be written in the form

$$\Phi(\zeta) + \frac{\omega(\zeta)}{\bar{\omega}'(\bar{\zeta})}\bar{\Phi}'(\bar{\zeta}) + \bar{\Psi}(\bar{\zeta}) = \text{const.} = K \tag{28}$$

on the circles c_1 and c_2 shown in Fig. 5.

We can express the analytic functions $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ in the Laurent series

$$\Phi_0(\zeta) = \sum_{n=-\infty}^{\infty} g_n\zeta^n, \qquad \Psi_0(\zeta) = \sum_{n=-\infty}^{\infty} h_n\zeta^n.$$

Since we are not concerned with finding the displacements in the beam, the indeterminacies in $\phi(z)$ and $\psi(z)$ mentioned in Sec. 4 permit us to choose $g_0 = h_0 = 0$, g_1 real. Because of the symmetry of the stress distribution in the cross section, the remaining coefficients in these series must be real, and only odd powers of ζ can appear. Hence we can write

$$\Phi_0(\zeta) = \sum_{n=-\infty}^{\infty} a_n \zeta^{2n+1}, \qquad \Psi_0(\zeta) = \sum_{n=-\infty}^{\infty} b_n \zeta^{2n+1}. \tag{29}$$

The real constants a_n and b_n must be chosen so that the functions $\Phi(\zeta)$ and $\Psi(\zeta)$, given by (26) and (27), satisfy the condition (28) on the circles c_1 and c_2.

Let us write

$$e^{i\theta} = \sigma.$$

Then on the outer circle c_1 we have

$$\zeta = \sigma, \qquad \bar{\zeta} = \frac{1}{\sigma}.$$

Thus (28) yields

$$\Phi(\sigma) + \frac{\omega(\sigma)}{\bar{\omega}'(1/\sigma)} \bar{\Phi}'\left(\frac{1}{\sigma}\right) + \bar{\Psi}\left(\frac{1}{\sigma}\right) = K.$$

We now substitute in this equation from (26), (27), and (29) and obtain an identity involving logarithmic terms and Laurent series. The logarithmic terms vanish if

$$K = \tfrac{1}{2} Pi \qquad \text{on upper half of } c_1,$$
$$K = \tfrac{3}{2} Pi \qquad \text{on lower half of } c_1;$$

the Laurent series vanish if

$$a_n - ma_{n-1} - m(2n+1)a_{-n-1} - (2n-1)a_{-n} + c_{-n-1} = 2\,\frac{1+m^2}{1-m}\,\delta_{0n} \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{30}$$

where δ_{0n} is a Kronecker delta, and

$$c_n = b_n - mb_{n+1}. \tag{31}$$

On the inner circle c_2, we have

$$\zeta = k\sigma, \qquad \bar{\zeta} = \frac{k}{\sigma},$$

so the boundary condition (28) yields

$$\Phi(k\sigma) + \frac{\omega(k\sigma)}{\bar{\omega}'(k/\sigma)} \bar{\Phi}'\left(\frac{k}{\sigma}\right) + \bar{\Psi}\left(\frac{k}{\sigma}\right) = K.$$

We again substitute in this equation from (26), (27), and (29) and obtain an identity involving Laurent series and terms involving log $(1 \pm k\sigma)$. Since $k < 1$, these logarithmic terms can be expanded in powers of σ. The coefficients of the various powers of σ in the above identity then vanish if $K = Pi$ and

$$(32)\quad k^{2n+1}a_n - mk^{2n-3}a_{n-1} - m(2n+1)k^{-2n-3}a_{-n-1} - (2n-1)k^{-2n+1}a_{-n} + k^{-2n-1}c_{-n-1} = 2p_n,$$

where c_n is as given in (31), and

$$p_n = k^{2n-3}\left[\frac{k^4}{2n+1} - \frac{m}{2n-1}\right] \qquad (n = 1, 2, \cdots),$$

$$(33)\quad p_0 = k + \frac{m}{k} + \frac{m + m^2 + k^2(1-m)}{(1-m)k},$$

$$p_{-n} = -k^{2n-1}\left[\frac{1}{2n-1} - \frac{m}{2n+1}\right] - (1-k^2)(k^2-m)k^{2m-3} \qquad (n = 1, 2, \cdots).$$

Equations (30) and (32) are to be solved for a_n and c_n. We first eliminate c_{-n-1} between these two equations, obtaining the relation

$$(34)\quad (1-k^{4n+2})a_n - (2n-1)(1-k^2)a_{-n} - m(1-k^{4n-2})a_{n-1} - m(2n+1)(1-k^{-2})a_{-n-1} = 2q_n,$$

where

$$q_n = -k^{4n-2}\left[\frac{k^4}{2n+1} - \frac{m}{2n-1}\right] \qquad (n = 1, 2, \cdots),$$

$$(35)\quad q_0 = 1 - 2k^2 - m,$$

$$q_{-n} = \frac{1}{2n-1} + 1 - k^2 - m\left[\frac{1}{2n+1} + \frac{1-k^2}{k^2}\right] \qquad (n = 1, 2, \cdots).$$

The infinite set of equations (34) are to be solved for the unknowns a_n. It will be recalled that the constant m in these equations is a measure of the eccentricity of the outer boundary of the cross section, while the constant k is a measure of the relative thickness of the walls of the beam. Once a_n have been determined, the unknown constants c_n are to be found from equations (30), and the unknown constants b_n are then to be determined from (31).

8. Determination of a_n. When $n = 0$, equation (34) yields one equation involving a_0, a_{-1}; when $n = \pm 1$, equation (34) yields two equations involving a_1, a_0, a_{-1}, a_{-2}; when $n = \pm 2$, equation (34) yields two equations involving a_2, a_1, a_{-2}, a_{-3}; and so on. It thus appears that there is no simple means of solving the infinite set of equations (34) successively. However, when $m = 0$, this set of equations can be solved successively. Since the case $m = 0$ is the case when the ellipses bounding the cross section degenerate into circles, and since m appears linearly in equation (34), it seems logical to attempt a solution of equation (34) by a perturbation method in which m is the perturbation parameter.

We first write equations (35) in the form

$$q_n = \sum_{p=0}^{\infty} q_n^{(p)} m^p, \tag{36}$$

where

$$\begin{aligned} q_n^{(0)} &= -\frac{k^{4n+2}}{2n+1} && (n = 1, 2, \cdots), \\ q_0^{(0)} &= 1 - 2k^2, && \\ q_{-n}^{(0)} &= \frac{1}{2n-1} + 1 - k^2 && (n = 1, 2, \cdots), \end{aligned} \tag{37}$$

$$\begin{aligned} q_n^{(1)} &= \frac{k^{4n-2}}{2n-1} && (n = 1, 2, \cdots), \\ q_0^{(1)} &= -1, && \\ q_{-n}^{(1)} &= -\frac{1}{2n+1} - \frac{1-k^2}{k^2} && (n = 1, 2, \cdots), \end{aligned} \tag{38}$$

$$q_n^{(p)} = 0 \qquad (p = 2, 3, \cdots). \tag{39}$$

We then seek a solution to equations (34) in the form

$$a_n = \sum_{p=0}^{\infty} a_n^{(p)} m^p, \tag{40}$$

where $a_n^{(p)}$ are independent of m. We now substitute in equation (34) for q_n and a_n from equations (36) and (40), and then equate to zero the coefficients of the various powers of m which result. This leads to the equation

$$\begin{aligned} (1 - k^{4n+2})a_n^{(p)} &- (2n-1)(1-k^2)a_{-n}^{(p)} \\ &= 2q_n^{(p)} + (1 - k^{4n-2})a_{n-1}^{(p-1)} + (2n+1)(1-k^{-2})a_{-n-1}^{(p-1)}. \end{aligned} \tag{41}$$

Equation (41) and the equation obtained from it by changing the sign of n can be solved readily for $a_n^{(p)}$, $a_{-n}^{(p)}$ in terms of $a_{n-1}^{(p-1)}$, $a_{-n-1}^{(p-1)}$. When $p = 0$, the solution is

$$a_n^{(0)} = \frac{Q_n^{(0)}}{D_n} \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{42}$$

where

$$\begin{aligned} D_n = D_{-n} &= -[(1 - k^{4n})^2 - 4n^2(1-k^2)^2 k^{4n-2}] \qquad (n = 0, 1, 2, \cdots), \\ Q_n^{(0)} &= \frac{2k^{4n-2}}{2n+1}[k^2(1 - k^{4n}) + 2n(1-k^2) + 4n^2(1-k^2)^2] \\ &\qquad\qquad (n = 1, 2, \cdots), \end{aligned} \tag{43}$$

$$\begin{aligned} Q_0^{(0)} &= -\frac{1-2k^2}{1-k^2}, \\ Q_{-n}^{(0)} &= \frac{2k^{4n-2}}{2n-1}[k^2(1-k^{4n}) + 2n(1-k^2)] \qquad (n = 1, 2, \cdots). \end{aligned} \tag{44}$$

When $p = 1$, the solution is

$$a_n^{(1)} = \frac{1}{D_n}\left[Q_n^{(1)} + R_n a_{n-1}^{(0)} + S_n a_{-n-1}^{(0)}\right] \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{45}$$

where D_n is given in (43) above, and

$$Q_n^{(1)} = -2k^{4n-2}\left[\frac{1 - k^{4n-2}}{2n - 1} + \frac{2n - 1}{2n + 1}(1 - k^2) + (2n - 1)(k^{-1} - k)^2\right] \qquad (n = 1, 2, \cdots),$$

$$\begin{aligned} Q_0^{(1)} &= \frac{1}{1 - k^2}, \\ Q_{-n}^{(1)} &= -2k^{4n-2}\left[(1 - k^{4n-2})\left(\frac{1}{2n + 1} + k^{-2} - 1\right) + \frac{2n + 1}{2n - 1}k^{4n-2}(1 - k^2)\right] \qquad (n = 1, 2, \cdots), \end{aligned} \tag{46}$$

$$\begin{aligned} R_n &= -[(1 - k^{4n-2})^2 - (2n - 1)^2(1 - k^2)^2 k^{4n-4}] \qquad (n = 1, 2, \cdots), \\ R_0 &= \tfrac{1}{2}k^{-2}, \\ R_{-n} &= k^{-4}R_{n+1} \qquad (n = 1, 2, \cdots), \end{aligned} \tag{47}$$

$$\begin{aligned} S_n &= \frac{1 - k^2}{k^4}[(1 + k^2)(1 - k^{4n}) - 2n(1 - k^2)(1 + k^{4n})] \qquad (n = 1, 2, \cdots), \\ S_0 &= \tfrac{1}{2}k^{-2}, \\ S_{-n} &= -k^{4n}S_n \qquad (n = 1, 2, \cdots). \end{aligned} \tag{48}$$

When $p = 2, 3, \cdots$, the solution is

$$a_n^{(p)} = \frac{1}{D_n}\left[R_n a_{n-1}^{(p-1)} + S_n a_{-n-1}^{(p-1)}\right] \quad (p = 2, 3, \cdots; n = 0, \pm 1, \pm 2, \cdots), \tag{49}$$

where D_n, R_n, and S_n are given in (43), (47), and (48), respectively.

9. Determination of b_n. To find the coefficients b_n, we use equations (30) and (31), together with the values of a_n determined as indicated in Sec. 8. This operation can be carried out conveniently within the framework of the perturbation method.

We first rewrite equation (30) in the form

$$\begin{aligned} c_n = -(2n + 3)a_{n+1} - a_{-n-1} - m(2n + 1)a_n + ma_{-n-2} \\ + 2\delta_{-1,n}\frac{1 + m^2}{1 - m} \qquad (n = 0, \pm 1, \pm 2, \cdots), \end{aligned} \tag{50}$$

where $\delta_{-1,n}$ is a Kronecker delta. We then express c_n in the form

$$c_n = \sum_{p=0}^{\infty} c_n^{(p)} m^p. \tag{51}$$

Since a_n has been expressed in the form

$$a_n = \sum_{p=0}^{\infty} a_n^{(p)} m^p,$$

and since

$$\frac{1}{1-m} = \sum_{p=0}^{\infty} m^p,$$

it follows from (50) that

$$c_n^{(p)} = -(2n+3)\,a_{n+1}^{(p)} - a_{-n-1}^{(p)} - (2n+1)\,a_n^{(p-1)} + a_{-n-2}^{(p-1)} + 4\delta_{-1,n} \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{52}$$

We now look for b_n in the form

$$b_n = \sum_{p=0}^{\infty} b_n^{(p)} m^p. \tag{53}$$

From (31) it then follows that

$$\begin{aligned} b_n^{(0)} &= c_n^{(0)} && (n = 0, \pm 1, \pm 2, \cdots), \\ b_n^{(p)} &= c_n^{(p)} + b_{n+1}^{(p-1)} && (n = 0, \pm 1, \pm 2, \cdots;\ p = 1, 2, 3, \cdots). \end{aligned} \tag{54}$$

The procedure for the determination of b_n is then the following: We first determine $c_n^{(p)}$ from equation (52) by using the values of $a_n^{(p)}$ found as indicated in Sec. 8. The quantities $b_n^{(p)}$ can then be found successively from equation (54), whence b_n can be found from equation (53).

10. Conclusions. As mentioned earlier, when $m = 0$, the ellipses bounding the cross section degenerate into concentric circles. In this case, the series expansions deduced above for $\Phi(\zeta)$ and $\Psi(\zeta)$ are easily seen to converge. When $m \neq 0$, a proof of convergence seems rather difficult to achieve. However, numerical computations carried out for the case when the relative sizes of the inner and outer ellipses bounding the cross section are such that $k = \frac{1}{2}$ indicates a convergence of these series in this case.

The constant m is a measure of the eccentricity of the ellipses bounding the cross section. Since m is also the parameter on which the perturbation method employed above is based, it follows that calculations yielding $\Phi(\zeta)$ and $\Psi(\zeta)$ in the case of a specific value of m can be extended readily to cases when m takes on other values.

It was mentioned in Sec. 3 that, in an attack on the present problem by the method of curvilinear coordinates, severe algebraic difficulties were encountered. Even if these difficulties could be surmounted, it would still be necessary to solve a set of recurrence relations for the coefficients in a series expansion for the stress function. Because of fundamental similarities between the method of curvilinear coordinates and the method of Muschelisvili, these recurrence relations should resemble those encountered in the present paper and should be no easier to solve.

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ON AXISYMMETRICAL DEFORMATIONS OF THIN SHELLS OF REVOLUTION[1]

BY

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1. Introduction. The following is concerned with the nonlinear theory of finite deflections of thin elastic shells. Attention is restricted to shells of revolution and axisymmetrical states of deformation. With these restrictions, the basic equations of the problem are a system of ordinary differential equations, at least for problems which do not involve time as one of the independent variables.

While a great deal of work has been done concerning the linear theory of small deflections of thin shells, relatively little is known concerning finite deflections of shells. We must however, mention the work of Donnell [3], Biezeno [1], von Kármán and Tsien [11, 12, 13] and of Friedrichs [5] on the subject of a nonlinear theory of elastic stability of shells and that of Bromberg and Stoker [2] on a nonlinear membrane theory of spherical shells subjected to internal pressure.

Our object in the present paper may be described as follows:

1. We formulate the problem of finite deflections in a manner as closely analogous as possible to the classical formulation of the corresponding small-deflection problem, namely, by means of two simultaneous second-order differential equations, relating to each other a basic deformation variable and a basic stress resultant variable. In this way we arrive at a formulation of the shell problem which is a natural extension of a part of the theory of finite deflections of thin rods, namely, that part which is concerned with the plane deformation of rods the center line of which is a plane curve.
2. On the basis of this formulation, we present criteria for the applicability of the linear small-deflection theory. It is to be expected that the range of applicability of linear shell theory is in some respects wider than that of the linear theory of bending of flat plates. This we confirm for the examples of the cylindrical shell and the spherical shell, in such a way that the results are significant also for other shells of revolution.
3. We discuss the problem of solution of the basic equations for the cylindrical shell and for the spherical shell for important specific problems. In so doing we obtain results for the spherical shell which are more comprehensive than those known heretofore. These results are such that the nature of their generalization for nonspherical shells of revolution can be foretold.

The present paper does not deal with the problem of the nonlinear theory of elastic stability of shells. However, the basic equations which we have derived should be helpful in permitting a more accurate analysis of this problem than has heretofore been given.

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2. Basic equations. The system of basic equations to be formulated contains (1) differential equations of equilibrium for elements of the shell, (2) stress-strain relations, (3) expressions for the components of strain in terms of appropriate deformation variables, and (4) a relevant equation of compatibility. These equations have been derived in detail elsewhere [8]. We list here the results in the following form:

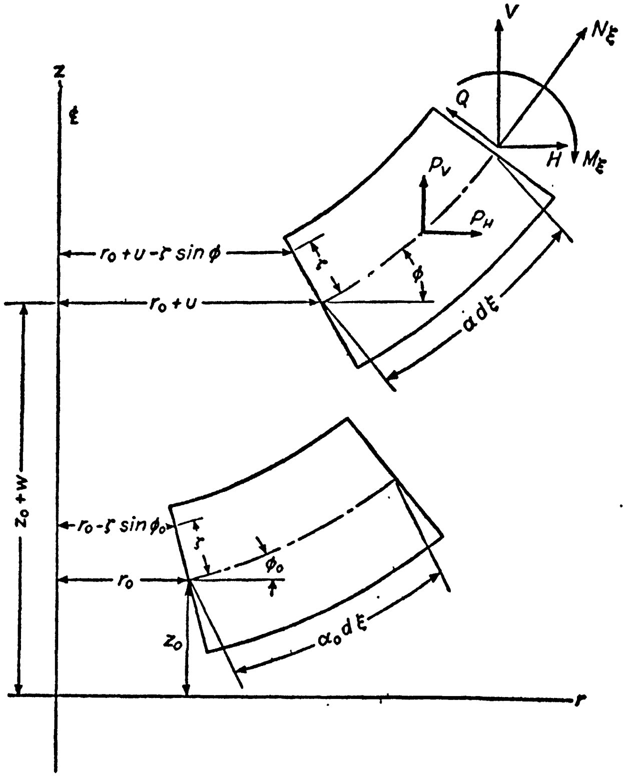

FIG. 1. Side view of element of shell in undeformed and in deformed state. Also shown are visible stress resultants and couples and load intensity components.

$$\begin{aligned} (rV)' + r\alpha p_V &= 0, \\ (rH)' - \alpha N_\theta + r\alpha p_H &= 0, \\ (rM_\xi)' - \alpha\cos\phi\, M_\theta - r\alpha Q &= 0. \end{aligned} \tag{1}$$

$$\begin{aligned} N_\xi &= H\cos\phi + V\sin\phi, \\ Q &= -H\sin\phi + V\cos\phi. \end{aligned} \tag{2}$$

$$\begin{aligned} C\epsilon_{\xi M} &= N_\xi - \nu N_\theta, & C\epsilon_{\theta M} &= N_\theta - \nu N_\xi, \\ M_\xi &= D(\kappa_\xi + \nu\kappa_\theta), & M_\theta &= D(\kappa_\theta + \nu\kappa_\xi). \end{aligned} \tag{3}$$

$$C = Eh, \qquad D = \frac{Eh^3}{12}(1-\nu^2). \tag{4}$$

$$\begin{aligned} \epsilon_{\xi M} &= \frac{\cos\phi_0}{\cos\phi}\left(1 + \frac{u'}{r_0'}\right) - 1, \qquad & \epsilon_{\theta M} &= \frac{u}{r_0}, \\ \kappa_\xi &= -\frac{\phi' - \phi_0'}{\alpha_0}, & \kappa_\theta &= -\frac{\sin\phi - \sin\phi_0}{r_0}. \end{aligned} \tag{5}$$

$$\cos\phi_0\,(r_0\epsilon_{\theta M})' - \cos\phi\,(r_0'\epsilon_{\xi M}) = r_0'(\cos\phi - \cos\phi_0). \tag{6}$$

$$u = r_0\epsilon_{\theta M}, \qquad w = \int \alpha_0[(1+\epsilon_{\xi M})\sin\phi - \sin\phi_0]\,d\xi. \tag{7}$$

$$\begin{aligned} \alpha_0^2 &= (r_0')^2 + (z_0')^2, \qquad r_0' = \alpha_0\cos\phi_0, \qquad z_0' = \alpha_0\sin\phi_0, \\ \alpha &= (1+\epsilon_{\xi M})\alpha_0, \qquad\qquad r = (1+\epsilon_{\theta M})r_0. \end{aligned} \tag{8}$$

Stress resultants V, H, N, Q and stress couples M occurring in equations (1) to (3) are defined in accordance with Figs. 1 and 2. The equation of the middle surface of the shell in its undeformed state is taken in the parametric form $r_0 = r_0(\xi)$, $z_0 = z_0(\xi)$. The equation of the middle surface in its deformed state is taken in the form $r = r_0 + u$, $z = z_0 + w$ so that u and w are components of

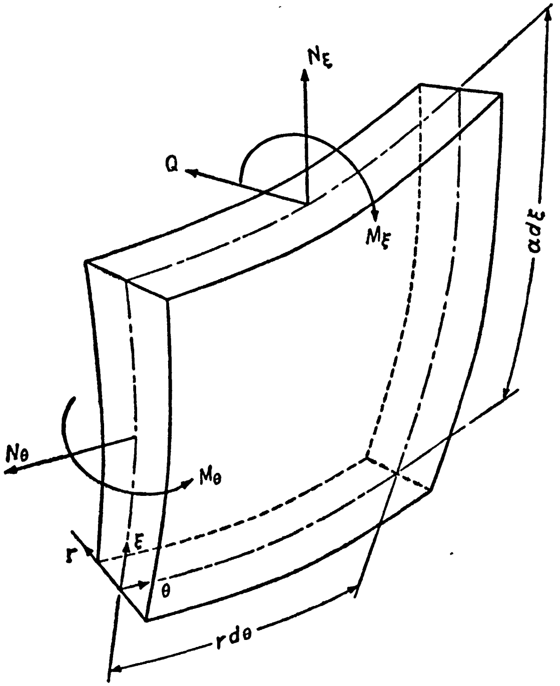

FIG. 2. Element of shell showing stress resultants and stress couples.

displacement in radial and in axial directions. Primes indicate differentiation with respect to the parameter ξ. The compatibility equation (6) is an immediate consequence of the definitions of the components of strain $\epsilon_{\xi M}$ and $\epsilon_{\theta M}$ in equation (5).

It should be noted that the state of strain in the shell is described by means of the radial displacement component u and the sloping angles ϕ and ϕ_0 of the meridians of the deformed and the undeformed middle surface, without reference to the axial displacement component w. Elimination of w from the expressions for strain is an important step in the derivation of the results which follow. One consequence of it is the ready derivation of the compatibility equation (6) from the first two of equations (5).

Expressions for the strains in accordance with equation (5) and for the displacement w in accordance with equation (7) are obtained by direct calculation on the basis of the geometry of Fig. 1. In this calculation, the assumption is made that normals to the undeformed middle surface are deformed, without extension, into normals to the deformed middle surface. This is the same assumption that is made in the small-deflection theory of shells and also in the finite deflection theory of thin rods. There is no reason to assume that it is less appropriate for the class of problem considered in this paper.

While we wish to admit large deformations of the shell, we wish at the same time to restrict attention to problems of small strain. In fact, if we did not do so, we could not put reliance on the stress-strain relations (3). In view of this restriction and in view of the last two of equations (8), we may in the equilibrium equations (1) replace r by r_0 and α by α_0 and use the equilibrium equations in the following modified form:

$$(1') \qquad \begin{aligned} &(r_0V)' + r_0\alpha_0 p_V = 0, \qquad (r_0H)' - \alpha_0 N_\theta + r_0\alpha_0 p_H = 0,\\ &(r_0M_\xi)' - \alpha_0 \cos\phi\, M_\theta - r_0\alpha_0(-H\sin\phi + V\cos\phi) = 0. \end{aligned}$$

We may note that equations (1′) may also be derived as the Euler equations of the variational problem to minimize the potential energy of the deformed shell, if the strain energy per unit of volume of the undeformed shell is assumed to be a quadratic expression in terms of the strains $\epsilon_{\xi M}$, $\epsilon_{\theta M}$, κ_ξ, and κ_θ, in the same way as in the theory of small deformations.

3. Reduction to two simultaneous equations. Evidently the system of the equilibrium equations (1′) and of the stress strain relations (3) is one of seven equations for the seven unknowns V, H, N_θ, M_ξ, M_θ, ϕ, and u. We reduce this system, in analogy to corresponding work in the small-deflection theory, to two simultaneous equations for the unknown meridian angle ϕ and a stress function Ψ defined in terms of the horizontal stress resultant H by

$$(9) \qquad \Psi = r_0 H.$$

We have in terms of ϕ and Ψ the following expressions for all other quantities which are needed:

$$\begin{aligned} r_0 V &= -\int r_0 p_V \alpha_0 \, d\xi, \\ r_0 N_\xi &= \Psi \cos\phi + (r_0 V) \sin\phi, \\ r_0 Q &= -\Psi \sin\phi + (r_0 V) \cos\phi, \\ \alpha_0 N_\theta &= \Psi' + \alpha_0 r_0 p_H. \end{aligned} \tag{10}$$

$$\begin{aligned} M_\xi &= -D\left[\frac{\phi' - \phi_0'}{\alpha_0} + \nu \frac{\sin\phi - \sin\phi_0}{r_0}\right], \\ M_\theta &= -D\left[\frac{\sin\phi - \sin\phi_0}{r_0} + \nu \frac{\phi' - \phi_0'}{\alpha_0}\right]. \end{aligned} \tag{11}$$

$$\begin{aligned} u &= \frac{r_0}{Eh}(N_\theta - \nu N_\xi), \\ w &= \int \left[\alpha_0 (\sin\phi - \sin\phi_0) + \frac{\alpha_0}{Eh}(N_\xi - \nu N_\theta) \sin\phi\right] d\xi. \end{aligned} \tag{12}$$

The two simultaneous equations for ϕ and Ψ are obtained by

1. Introducing M_ξ and M_θ from equations (11) into the third of the equilibrium equations (1′),
2. Introducing $\epsilon_{\xi M}$ and $\epsilon_{\theta M}$, as expressed in terms of ϕ and Ψ by means of equations (3) and (10), into the compatibility equation (6).

The resultant two differential equations may be written in various ways. We here write

$$\begin{aligned} \text{(I)}\quad & (\phi'' - \phi_0'') + \frac{(r_0 D/\alpha_0)'}{(r_0 D/\alpha_0)}(\phi' - \phi_0') - \left(\frac{\alpha_0}{r_0}\right)^2 \cos\phi\,(\sin\phi - \sin\phi_0) \\ & + \nu \frac{\alpha_0}{r_0}\left[(\cos\phi - \cos\phi_0)\phi_0' + \frac{D'}{D}(\sin\phi - \sin\phi_0)\right] \\ & = \frac{\alpha_0^2}{r_0 D}[\Psi \sin\phi - (r_0 V)\cos\phi]. \end{aligned}$$

$$\begin{aligned} \text{(II)}\quad & \Psi'' + \frac{(r_0/C\alpha_0)'}{(r_0/C\alpha_0)}\Psi' - \left[\left(\frac{\alpha_0}{r_0}\right)^2 \cos^2\phi - \nu \frac{\alpha_0}{r_0}\left(\phi' \sin\phi + \frac{C'}{C}\cos\phi\right)\right]\Psi \\ & = \frac{\alpha_0^2 C}{r_0}(\cos\phi - \cos\phi_0) + \nu \frac{\alpha_0}{r_0} \sin\phi\,(r_0 V)' \\ & + \left[\left(\frac{\alpha_0}{r_0}\right)^2 \cos\phi \sin\phi + \nu \frac{\alpha_0}{r_0}\left(\phi' \cos\phi - \frac{C'}{C}\sin\phi\right)\right](r_0 V) \\ & - \frac{\alpha_0}{r_0}(r_0^2 p_H)' - \left[\nu \left(\frac{\alpha_0}{r_0}\right)^2 \cos\phi - \frac{\alpha_0}{r_0}\frac{C'}{C}\right](r_0^2 p_H). \end{aligned}$$

Equations (I) and (II) as they stand are valid for arbitrarily large deformations, *i.e.*, for arbitrarily large values of the difference $\phi - \phi_0$ of the tangent angles. Linearization reduces them to a form equivalent to the basic results of H. Reissner [10] and E. Meissner [6].

For many purposes it is sufficient to restrict attention to small finite angle changes. The appropriate system of equations will be listed in the following section.

4. The equations of the theory of small finite deflections. We write

$$\beta = -(\phi - \phi_0) \tag{13}$$

and retain in equations (10) to (12) and (I) and (II) terms up to the second degree in β and Ψ. Since

$$\begin{aligned} \cos\phi &= \cos\phi_0 + \beta\sin\phi_0 - \tfrac{1}{2}\beta^2\cos\phi_0 + \cdots, \\ \sin\phi &= \sin\phi_0 - \beta\cos\phi_0 - \tfrac{1}{2}\beta^2\sin\phi_0 + \cdots, \end{aligned} \tag{14}$$

we obtain from equations (10) and (11)

$$\begin{aligned} r_0V &= -\int r_0p_V\alpha_0\,d\xi, \\ r_0N_\xi &= \Psi\cos\phi_0 + (r_0V)\sin\phi_0 + \beta[\Psi\sin\phi_0 - (r_0V)\cos\phi_0], \\ r_0Q &= -\Psi\sin\phi_0 + (r_0V)\cos\phi_0 + \beta[\Psi\cos\phi_0 + (r_0V)\sin\phi_0], \\ \alpha_0N_\theta &= \Psi' + r_0\alpha_0p_H. \end{aligned} \tag{15}$$

$$\begin{aligned} M_\xi &= D\left[\frac{\beta'}{\alpha_0} + \nu\left(\frac{\cos\phi_0}{r_0}\beta + \frac{1}{2}\frac{\sin\phi_0}{r_0}\beta^2\right)\right], \\ M_\theta &= D\left[\frac{\cos\phi_0}{r_0}\beta + \frac{1}{2}\frac{\sin\phi_0}{r_0}\beta^2 + \nu\frac{\beta'}{\alpha_0}\right]. \end{aligned} \tag{16}$$

We omit listing the corresponding expressions for u and w which follow from (12).

The differential equations (I) and (II) assume now the following form:

$$\begin{aligned} &\beta'' + \frac{(r_0D/\alpha_0)'}{(r_0D/\alpha_0)}\beta' - \left[\left(\frac{r_0'}{r_0}\right)^2 - \nu\frac{(r_0'D/\alpha_0)'}{(r_0D/\alpha_0)}\right]\beta \\ &\quad - \left[\frac{3}{2}\frac{r_0'z_0'}{r_0^2} - \frac{\nu}{2}\frac{(z_0'D/\alpha_0)'}{(r_0D/\alpha_0)}\right]\beta^2 \\ &\quad = -\frac{\alpha_0^2}{r_0D}[\Psi\sin\phi_0 - (r_0V)\cos\phi_0 - \beta\{\Psi\cos\phi_0 + (r_0V)\sin\phi_0\}]. \end{aligned} \tag{III}$$

$$\begin{aligned} &\Psi'' + \frac{(r_0/C\alpha_0)'}{(r_0/C\alpha_0)}\Psi' - \left[\left(\frac{r_0'}{r_0}\right)^2 + \nu\frac{(r_0'/C\alpha_0)'}{(r_0/C\alpha_0)}\right]\Psi \\ &\quad - \left[2\frac{z_0'r_0'}{r_0^2} + \nu\frac{(z_0'/C\alpha_0)'}{(r_0/C\alpha_0)}\right]\beta\Psi - \nu\frac{z_0'}{r_0}\beta'\Psi = \frac{\alpha_0^2C}{r_0}[\beta\sin\phi_0 - \tfrac{1}{2}\beta^2\cos\phi_0] \\ &\quad + \left[\frac{r_0'z_0'}{r_0^2} + \nu\frac{(z_0'/C\alpha_0)'}{(r_0/C\alpha_0)}\right](r_0V) + \nu\frac{z_0'}{r_0}(r_0V)' \\ &\quad + \left[\frac{(z_0')^2 - (r_0')^2}{r_0^2} - \nu\frac{(r_0'/C\alpha_0)'}{(r_0/C\alpha_0)}\right]\beta(r_0V) - \nu\frac{r_0'}{r_0}\beta'(r_0V) \\ &\quad - \nu\frac{r_0'}{r_0}\beta(r_0V)' - \frac{\alpha_0}{r_0}(r_0^2p_H)' - \left[\nu\left(\frac{r_0'\alpha_0}{r_0^2} + \beta\frac{z_0'\alpha_0}{r_0^2}\right) - \frac{\alpha_0}{r_0}\frac{C'}{C}\right](r_0^2p_H). \end{aligned} \tag{IV}$$

It will be found that in most cases a number of the terms in equations (III) and (IV), namely, the nonlinear terms on the left of these equations and the terms containing (r_0V) on the right of equation (IV), are negligibly small. The most essential nonlinear term is in all cases the term with β on the right of equation (III). The term with β^2 on the right of equation (IV) is essential whenever $\sin\phi_0$ is small compared with $\cos\phi_0$, *i.e.*, in the shallow regions of a shell.

5. The equations for bending of a flat plate. Equations for finite deflections of flat plates are obtained by setting

$$(17) \qquad r_0 = \xi, \qquad z_0 = 0, \qquad \alpha_0 = 1, \qquad \sin\phi_0 = 0, \qquad \cos\phi_0 = 1.$$

Stress resultants, stress couples, and displacements follow from equations which are the appropriate special cases of equations (10) to (12). The differential equations (I) and (II) reduce to the following relations:

$$(\mathrm{V}) \quad \phi'' + \frac{(\xi D)'}{(\xi D)}\phi' - \frac{\cos\phi\sin\phi}{\xi^2} + \frac{\nu D'}{\xi D}\sin\phi = \frac{1}{\xi D}[\Psi\sin\phi - (r_0V)\cos\phi].$$

$$\begin{aligned}(\mathrm{VI}) \quad \Psi'' + \frac{(\xi/C)'}{(\xi/C)}\Psi' - \left[\frac{\cos^2\phi}{\xi^2} - \frac{\nu}{\xi}\left(\phi'\sin\phi + \frac{C'}{C}\cos\phi\right)\right]\Psi \\ = \frac{C}{\xi}(\cos\phi - 1) - \frac{(\xi^2p_H)'}{\xi} - \left[\frac{\nu\cos\phi}{\xi^2} - \frac{C'}{C\xi}\right](\xi^2p_H) \\ + \left[\frac{\sin\phi\cos\phi}{\xi^2} + \frac{\nu}{\xi}\left(\phi'\cos\phi - \frac{C'}{C}\sin\phi\right)\right](\xi V) + \frac{\nu\sin\phi}{\xi}(\xi V)'.\end{aligned}$$

When $C' = D' = 0$, *i.e.*, for plate of uniform thickness, these equations reduce to results which have been given at an earlier occasion [9]. In particular, for a plate of uniform thickness, acted upon by edge moments and horizontal edge loads only, they reduce to the following form:

$$(\mathrm{V}') \qquad D[\xi^2\phi'' + \xi\phi' - \cos\phi\sin\phi] = \xi\Psi\sin\phi.$$

$$(\mathrm{VI}') \qquad \xi^2\Psi'' + \xi\Psi' - (\cos^2\phi - \nu\xi\phi'\sin\phi)\Psi = C\xi(\cos\phi - 1).$$

As this account is primarily concerned with shells, nothing further will here be said concerning the subject of originally flat plates.

6. The equations for finite deflections of cylindrical shells of uniform thickness. These equations are obtained by setting

$$(18) \qquad r_0 = a, \qquad z_0 = a\xi, \qquad \alpha_0 = a, \qquad \sin\phi_0 = 1, \qquad \cos\phi_0 = 0.$$

We further have, in accordance with (13),

$$(19) \qquad \phi = \tfrac{1}{2}\pi - \beta, \qquad \cos\phi = \sin\beta, \qquad \sin\phi = \cos\beta,$$

so that the basic equations (I) and (II) reduce to the following:

$$(20a) \qquad D[\beta'' - (1 - \cos\beta)\sin\beta] = a[-\Psi\cos\beta + (aV)\sin\beta],$$

$$\begin{aligned}(20b) \quad \Psi'' - (\sin^2\beta + \nu\beta'\cos\beta)\Psi = aC\sin\beta - a^2p_H' - \nu\sin\beta a^2p_H \\ + (\cos\beta - \nu\beta')\sin\beta\,(aV) + \nu\cos\beta\,(aV)'.\end{aligned}$$

Equations (10) to (12) for resultants, couples, and displacements become

$$\begin{aligned} V &= -a\int p_V\,d\xi, \\ aN_\xi &= \Psi\sin\beta + (aV)\cos\beta, \\ aQ &= -\Psi\cos\beta + (aV)\sin\beta, \\ aN_\theta &= \Psi' + a^2p_H. \end{aligned} \tag{21}$$

$$\begin{aligned} aM_\xi &= D[\beta' + \nu(1-\cos\beta)], \\ aM_\theta &= D[(1-\cos\beta) + \nu\beta']. \end{aligned} \tag{22}$$

$$\begin{aligned} Ehu &= \Psi' + a^2p_H - \nu(\Psi\sin\beta + aV\cos\beta), \\ Ehw &= \int [-a(1-\cos\beta)Eh + \cos\beta\,(\Psi\sin\beta \\ &\qquad + aV\cos\beta - \nu\Psi' - \nu a^2p_H)]\,d\xi. \end{aligned} \tag{23}$$

The equations of finite small deflections are obtained from equations (19) and (20) by retaining terms up to the second degree in the unknowns. They are

$$D\beta'' = a[-\Psi + aV\beta], \tag{24}$$

$$\Psi'' - \nu\beta'\Psi = aC\beta + aV\beta + \nu aV' - a^2p_H' - \nu\beta a^2p_H. \tag{25}$$

Note that, except for the term $\nu\beta'\Psi$, these equations are linear in the variables β and Ψ as long as V, in accordance with the first of equations (21), is independent of β and Ψ. It turns out that the one nonlinear term in (25) is always negligible. Furthermore, it is readily seen that, as long as the values of the elastic stresses are small compared with the value of the modulus of elasticity E, the second term and the last term on the right of equation (25) are always negligible compared with the first term. Consequently, equation (25) may be further simplified to read

$$\Psi'' = aC\beta + \nu aV' - a^2p_H'. \tag{25'}$$

While equations (24) and (25′) are linear in appearance, they do describe nonlinear as well as linear phenomena as will be shown.

7. Cylindrical shell acted upon by vertical edge loads (Fig. 3). It is assumed that no surface loads are acting, *i.e.*, we set $p_V = p_H = 0$, and that the following boundary conditions are prescribed:

$$\xi = \pm\xi_e, \quad u = 0, \quad M_\xi = 0, \quad V = P. \tag{26}$$

Positive values of P mean that the shell is in axial tension while negative values of P mean axial compression.

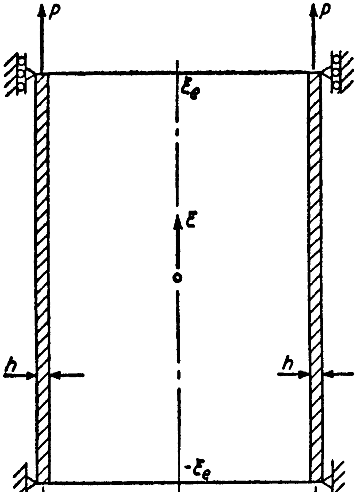

FIG. 3

The differential equations (24) and (25′) become

$$\begin{aligned} D\beta'' + a\Psi &= a^2 P\beta, \\ \Psi'' - aC\beta &= 0. \end{aligned} \tag{27}$$

The boundary conditions (26) may, for small finite deflections, be written in the form

$$\xi = \pm\xi_e, \qquad \Psi' = \nu a P, \qquad \Psi''' = 0. \tag{28}$$

In writing the second of the conditions (28), the quantity M_ξ, as given by equation (22), has been transformed by the use of the second of equations (27). The two equations (27) may be reduced to the following single equation:

$$\frac{D}{a^2}\frac{d^4\Psi}{d\xi^4} - P\frac{d^2\Psi}{d\xi^2} + C\Psi = 0. \tag{29}$$

The problem in this form is nonlinear because of the way in which the parameter P occurs both in the differential equation (29) and in one of the boundary conditions (28).

The buckling problem corresponding to negative values of P has been discussed by L. Föppl [4] and will not be considered here. Our concern is with the problem for positive values of P and the classification of various situations that may arise.

We know that for sufficiently small values of P the term with P in the differential equation (29) is negligible. We say that in that range of values of P where this is the case we have a problem of *linear bending*.

On the other hand, if we disregard the condition that $\Psi'''(\pm\xi_e) = 0$, we also know that for sufficiently large values of P the fourth-derivative term in equation (29) is negligible. When this is the case, we may call the problem, in analogy to what has been done by Bromberg and Stoker [2] for a problem of spherical-shell deformation which we will discuss later, a problem of the *nonlinear membrane* theory. Note that in applying this theory we presuppose that neglect of the bending stresses compared with the direct stresses, which is implied by disregarding the third-derivative boundary condition, can subsequently be justified on the basis of the solution obtained.

Evidently besides these two states of linear bending and nonlinear membrane action, there will be an in-between state where both bending and nonlinear membrane action will have to be taken into account.

What we propose to do next is to delimit the range of appropriate parameter values for a problem to fall in one of three classes mentioned. While for the present problem this can be done by a discussion of the explicit solution of the boundary-value problem, we may accomplish the same in the following way. Taking account of the fact that the nonuniformity of the state of stress in the shell will occur in edge zones adjacent to the boundaries $\xi = \pm\xi_e$ and be identical in character at both edges, we introduce new variables η and ψ in accordance with the following definitions:

$$\xi_e - \xi = \lambda\eta, \qquad \Psi = aP\lambda\,\psi(\eta). \tag{30}$$

The boundary conditions (28) for $\xi = +\xi_e$ may then be written in the form

$$\eta = 0; \qquad \frac{d\psi}{d\eta} = -\nu, \qquad \frac{d^3\psi}{d\eta^3} = 0, \tag{31}$$

while the differential equation (29) may be written in the form

$$\frac{D}{a^2C\lambda^4}\frac{d^4\psi}{d\eta^4} - \frac{P}{C\lambda^2}\frac{d^2\psi}{d\eta^2} + \psi = 0. \tag{32}$$

When bending is important, the term with $d^4\psi/d\eta^4$ must be of the same order of magnitude as the term with ψ, and so we may set

$$\frac{D}{a^2C\lambda^4} = 1, \qquad \lambda = \sqrt[4]{12(1-\nu^2)}\sqrt{\frac{h}{a}}. \tag{33}$$

This makes equation (32) read

$$\frac{d^4\psi}{d\eta^4} - \frac{Pa}{\sqrt{CD}}\frac{d^2\psi}{d\eta^2} + \psi = 0. \tag{34}$$

Equation (34) is appropriate as long as the coefficient $Pa/\sqrt{CD}$ is not large compared with unity. If we write $P = \sigma_D h$, this coefficient may be expressed as follows:

$$\frac{Pa}{\sqrt{CD}} = \sqrt{12(1-\nu^2)}\,\frac{\sigma_D}{E}\frac{a}{h}. \tag{35}$$

We now distinguish two cases.

Case 1

$$\frac{Pa}{\sqrt{CD}} \ll 1. \tag{36}$$

When the order of magnitude relation (36) holds, the effect of axial tension is negligible, and we have a problem of linear bending.

Case 2

$$\frac{Pa}{\sqrt{CD}} = O(1). \tag{37}$$

In this case, linear bending and nonlinear membrane action both have to be taken into account.

We note that the order of magnitude of the tensile stress σ_D, for case 2, is the same as the known buckling stress when σ_D is negative and that, for case 1, σ_D is small compared to the value of this buckling stress.

Returning to the differential equation (32), we obtain the remaining case 3 by setting

$$\frac{P}{C\lambda^2} = 1, \qquad \lambda = \sqrt{\frac{P}{C}} = \sqrt{\frac{\sigma_D}{E}}. \tag{38}$$

Equation (32) now becomes

$$\frac{DC}{a^2P^2}\frac{d^4\psi}{d\eta^4} - \frac{d^2\psi}{d\eta^2} + \psi = 0. \tag{39}$$

The fourth-derivative term is negligible, and we are in the range of the nonlinear membrane theory when $DC/a^2P^2 \ll 1$, and this condition may also be written in the following form:

Case 3

$$\frac{Pa}{\sqrt{CD}} \gg 1. \tag{40}$$

Equations (36), (37), and (40) cover all cases that may occur in conjunction with the differential equation (32).

It remains to show (1) that the bending stresses σ_B are in fact negligible compared with σ_D when case 3 is given and (2) to what extent it is permissible to omit the nonlinear β terms in equations (19) and (20) when dealing with the nonlinear membrane state.

To answer the first question, we note that $\sigma_B = (h/2)(E/a)(d\beta/d\xi)$ and, in view of the first of equations (27),

$$\sigma_B = \frac{Eh}{2a^2P}\frac{d\Psi}{d\xi} = \frac{hE}{2a}\frac{d\psi}{d\eta} = O\left(E\frac{h}{a}\right). \tag{41}$$

Comparison of equation (40), which in view of equation (35) is equivalent to the condition $\sigma_D \gg E(h/a)$, and equation (41) shows that, when case 3 is given, then $\sigma_B \ll \sigma_D$, which is as it should be.

To answer the second question concerning the omission of the nonlinear β terms, we observe that

$$\beta = \frac{1}{aP}\Psi = \lambda\psi = \sqrt{\frac{\sigma_D}{E}}\,\psi = O\left(\sqrt{\frac{\sigma_D}{E}}\right). \tag{42}$$

As long as we consider problems involving small strain only, we have $\sigma_D/E \ll 1$, and it is then justified in view of (42) to omit the nonlinear β terms in the differential equations of the present problem of axial tension.

We may finally note that the range of applicability of the linear bending theory (case 1) is such that in it the radial deflection u is small compared with the shell thickness h. We shall show in the next section that there are other problems in shell theory where a different length parameter determines the range of applicability of the small deflection theory.

8. Cylindrical shell acted upon by edge bending moments (Fig. 4). We consider a semiinfinite shell acted upon by bending moments M_0 at the edge $\xi = 0$. Since for this problem we have $V = p_H = 0$, the differential equations (19) and (20) reduce to the following:

$$\begin{aligned} D[\beta'' - (1 - \cos\beta)\sin\beta] &= -a\Psi\cos\beta, \\ \Psi'' - (\sin^2\beta + \nu\beta'\cos\beta)\Psi &= aC\sin\beta. \end{aligned} \tag{43}$$

The boundary conditions are

$$(44)\qquad \begin{aligned} aM_\xi(0) &= D[\beta'(0) + \nu\{1 - \cos\beta(0)\}] = aM_0, \\ aH(0) &= \Psi(0) = 0, \end{aligned}$$

together with the condition that stresses and displacements vanish for $\xi = \infty$.

It is readily seen that the nonlinear terms on the left of equations (43) and (44) are always negligible, and so we may restrict attention to the following system of equations:

$$(45)\qquad D\beta'' = -a\Psi\cos\beta, \qquad \Psi'' = aC\sin\beta.$$

Correspondingly, the first of the boundary conditions (44) may be simplified by omission of the term multiplied by ν, and so the boundary conditions for the system (45) may be written in the form

$$(46)\qquad \beta'(0) = \frac{aM_0}{D} = 2(1-\nu^2)\frac{\sigma_0}{E}\frac{a}{h}, \qquad \Psi(0) = 0.$$

Evidently, the condition for linearity is that the sloping angle β satisfy the order of magnitude relation

$$(47)\qquad \beta^2 \ll 1.$$

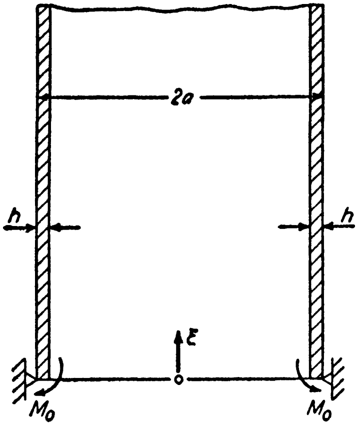

FIG. 4

When this is the case, equations (45) and (46) may be reduced to the following:

$$(48)\qquad \begin{gathered} \Psi^{IV} + \frac{a^2C}{D}\Psi = 0, \\ \Psi'''(0) = 2(1-\nu^2)\sigma_0 a^2, \qquad \Psi(0) = 0. \end{gathered}$$

In (48) we introduce new dependent and independent variables η and ψ by means of the relations

$$(49)\qquad \xi = \lambda\eta, \qquad \Psi = 2(1-\nu^2)\sigma_0 a^2\lambda^3\psi(\eta).$$

In order to make the resultant system parameter-free, we set

$$(50)\qquad \frac{1}{\lambda^4} = \frac{a^2C}{D}, \qquad \lambda = \frac{1}{\sqrt[4]{12(1-\nu^2)}}\sqrt{\frac{h}{a}}.$$

This changes the system (48) to

$$(51)\qquad \begin{gathered} \frac{d^4\psi}{d\eta^4} + \psi = 0, \\ \psi(0) = 0, \qquad \left(\frac{d^3\psi}{d\eta^3}\right)_0 = 1. \end{gathered}$$

Now

$$\beta = \frac{1}{aC}\frac{d^2\Psi}{d\xi^2} = \frac{2(1-\nu^2)\sigma_0 a^2\lambda^3}{aC\lambda^2}\frac{d^2\psi}{d\eta^2} = \frac{2(1-\nu^2)}{\sqrt[4]{12(1-\nu^2)}}\frac{\sigma_0}{E}\sqrt{\frac{a}{h}}\frac{d^2\psi}{d\eta^2}. \tag{52}$$

Since in view of (51) we have $d^2\psi/d\eta^2 = O(1)$, it follows that the condition (47) for linearity may be written in the form

$$\left(\frac{\sigma_0}{E}\right)^2\frac{a}{h} \ll 1. \tag{53}$$

In order to interpret this condition with reference to the radial deflection u, we take u from equation (23) and transform this expression as follows:

$$u = \frac{1}{Eh}\frac{d\Psi}{d\xi} = \frac{2(1-\nu^2)\sigma_0 a^2\lambda^3}{Eh\lambda}\frac{d\psi}{d\eta} = \frac{2(1-\nu^2)}{\sqrt{12(1-\nu^2)}}\frac{\sigma_0}{E}a\frac{d\psi}{d\eta}. \tag{54}$$

We conclude from equation (54) that u is of the order $(\sigma_0/E)a$. Since the condition for linearity (53) may be written in the form $\sigma_0/E \ll (h/a)^{\frac{1}{2}}$, it follows that the condition for linearity expressed as a restriction on u, is of the form

$$|\,u\,| \ll \sqrt{ha}. \tag{55}$$

Thus, *linearity in this problem does not require that the displacements be small compared with the thickness of the shell but rather that they be small compared with the geometric mean of the thickness and of the radius of the shell.*

It may be remarked that the same result is obtained if, instead of boundary conditions (44), we prescribe the value of the horizontal edge load $H(0)$ to have a value H_0 while $M_\xi(0)$ is assumed to vanish.

9. Differential equations for finite deflections of spherical shells of uniform thickness. We obtain the equations for a spherical shell of radius a by setting

$$r_0 = a\sin\xi, \qquad z_0 = -a\cos\xi, \qquad \alpha_0 = a, \qquad \phi_0 = \xi. \tag{56}$$

Direct and bending strains as defined by equations (5) become

$$\begin{aligned} \epsilon_{\xi M} &= \frac{\cos\xi}{\cos\phi} + \frac{u'}{a\cos\phi} - 1, \qquad \epsilon_{\theta M} = \frac{u}{a\sin\xi}, \\ \kappa_\xi &= \frac{1-\phi'}{a}, \qquad \kappa_\theta = \frac{1}{a}\left(1 - \frac{\sin\phi}{\sin\xi}\right). \end{aligned} \tag{57}$$

The two simultaneous differential equations (I) and (II) reduce to the following equations:

$$\begin{aligned} \phi'' + \cot\xi\,(\phi' - 1) - \cot^2\xi\,\frac{\cos\phi}{\cos\xi}\,\frac{\sin\phi - \sin\xi}{\cos\xi} + \nu\,\frac{\cos\phi - \cos\xi}{\sin\xi} \\ = \frac{a}{D}\left[\Psi\frac{\sin\phi}{\sin\xi} - r_0V\frac{\cos\phi}{\sin\xi}\right], \end{aligned} \tag{58}$$

$$\begin{aligned} \Psi'' + \cot\xi\,\Psi' - \left[\cot^2\xi\,\frac{\cos^2\phi}{\cos^2\xi} - \nu\phi'\frac{\sin\phi}{\sin\xi}\right]\Psi \\ = aC\,\frac{\cos\phi - \cos\xi}{\sin\xi} + \cot\xi\left[\frac{\cos\phi}{\cos\xi}\,\frac{\sin\phi}{\sin\xi} + \nu\phi'\frac{\cos\phi}{\cos\xi}\right](r_0V) \\ + \nu\,\frac{\sin\phi}{\sin\xi}(r_0V)' - \frac{(r_0^2p_H)'}{\sin\xi} - \nu\cot\xi\,\frac{\cos\phi}{\cos\xi}\,\frac{r_0^2p_H}{\sin\xi}. \end{aligned} \tag{59}$$

Equations (57) to (59) are put on record here with a view to future applications. In what follows we shall restrict attention to the differential equations for small finite deflections which are obtained by specialization of the general equations (III) and (IV). With r_0, z_0, α_0, and ϕ_0 from (56), and when D and C are constant, equations (III) and (IV) become

$$\beta'' + \cot\xi\,\beta' - (\cot^2\xi + \nu)\beta - \tfrac{1}{2}(3 - \nu)\cot\xi\,\beta^2 = -\frac{a}{D}[\Psi - (r_0V)\cot\xi - \beta(r_0V + \cot\xi\,\Psi)], \tag{60}$$

and

$$\begin{aligned}\Psi'' + \cot\xi\,\Psi' - (\cot^2\xi - \nu)\Psi - [(2 + \nu)\cot\xi\,\beta + \nu\beta']\Psi &= aC(\beta - \tfrac{1}{2}\cot\xi\,\beta^2) \\ &\quad + \nu(1 - \beta\cot\xi)(r_0V)' - \frac{(r_0^2p_H)'}{\sin\xi} - \nu(\beta + \cot\xi)\frac{(r_0^2p_H)}{\sin\xi} \\ &\quad + [(1 + \nu)\cot\xi + (1 + \nu - \cot^2\xi)\beta - \nu\beta'\cot\xi](r_0V).\end{aligned} \tag{61}$$

In terms of the solutions of these two differential equations, we have the following expressions for stress resultants and couples, in accordance with equations (15) and (16):

$$\begin{aligned} aN_\xi &= \Psi\cot\xi + (r_0V) + \beta[\Psi - (r_0V)\cot\xi], \\ aQ &= -\Psi + (r_0V)\cot\xi + \beta[\Psi\cot\xi + (r_0V)], \\ aN_\theta &= \Psi' + ar_0p_H, \\ aM_\xi &= D[\beta' + \nu\cot\xi\,\beta + \tfrac{1}{2}\nu\beta^2], \\ aM_\theta &= D[\cot\xi\,\beta + \tfrac{1}{2}\beta^2 + \nu\beta']. \end{aligned} \tag{62}$$

Radial and axial displacements are given in the form

$$\begin{aligned} Ehu &= a\sin\xi\,(N_\theta - \nu N_\xi), \\ Ehw &= -a\int[Eh(\cos\xi\,\beta + \tfrac{1}{2}\sin\xi\,\beta^2) - (\sin\xi - \beta\cos\xi)(N_\xi - \nu N_\theta)]\,d\xi. \end{aligned} \tag{63}$$

Various terms in equations (60) to (63) will turn out to be negligibly small in the class of problems which we consider in what follows.

10. Separation of effect of uniform normal pressure. In this section we give the form which the differential equations (60) and (61) assume, if in them we consider separately the effect of a uniform normal pressure p, such that $p_V = -p\sin\phi$, $p_H = p\cos\phi$, and of arbitrary additional components p_V and p_H. We write

$$\begin{aligned} r_0V &= -\int r_0\alpha_0p_V\,d\xi = \tfrac{1}{2}pa^2\sin^2\xi + P_V, \\ r_0^2p_H &= pa^2\sin^2\xi\,[\sin\xi - \beta\cos\xi] + P_H. \end{aligned} \tag{64}$$

In equations (64), P_V and P_H are the values of r_0V and $r_0^2p_H$ when $p = 0$.

Evidently, when $P_V = P_H = 0$, we have one solution for which $\beta = 0$ and for which then $\Psi = \frac{1}{2}pa^2\cos\xi\sin\xi$, and, according to (62), $N_\xi = N_\theta = \frac{1}{2}pa$.

We write for the solution of the complete system

$$\Psi = \tfrac{1}{2}pa^2\cos\xi\sin\xi + \psi. \tag{65}$$

With equations (64) and (65), we may rewrite the differential equations (60) and (61), after some calculations, in the following form:

$$\begin{aligned}(66)\quad & D[\beta'' + \cot\xi\,\beta' - (\cot^2\xi + \nu)\beta - \tfrac{1}{2}(3 - \nu)\cot\xi\,\beta^2] \\ &\qquad = -a[\psi - \tfrac{1}{2}\rho a^2\beta - P_V\cot\xi - (\cot\xi\,\psi + P_V)\beta],\end{aligned}$$

$$\begin{aligned}(67)\quad & \psi'' + \cot\xi\,\psi' - (\cot^2\xi - \nu)\psi - [(2+\nu)\cot\xi\,\beta + \nu\beta']\psi = aC(\beta - \tfrac{1}{2}\cot\xi\,\beta^2) \\ & + \nu(1 - \beta\cot\xi)P_V' + [(1+\nu)\cot\xi + (1 + \nu - \cot^2\xi)\beta - \nu\cot\xi\,\beta']P_V \\ & - \frac{P_H'}{\sin\xi} - \nu(\cot\xi + \beta)\frac{P_H}{\sin\xi} \\ &\qquad + \rho a^2\{\beta[(3+\nu)\cos^2\xi - \tfrac{1}{2}(1+\nu)] + \beta'\cos\xi\sin\xi\}.\end{aligned}$$

Corresponding expressions for stress resultants follow from (62), (64), and (65) in the form

$$(68)\qquad \begin{aligned} aN_\xi &= \tfrac{1}{2}\rho a^2 + \psi\cot\xi + P_V + (\psi - P_V\cot\xi)\beta, \\ aQ &= -\Psi + P_V\cot\xi + (\tfrac{1}{2}\rho a^2 + \psi\cot\xi + P_V)\beta, \\ aN_\theta &= \tfrac{1}{2}\rho a^2 + \psi' + \frac{P_H}{\sin\xi} - \rho a^2\beta\sin\xi\cos\xi. \end{aligned}$$

The couples M_ξ and M_θ remain given by (62). The horizontal component of displacement u as defined by (63) becomes

$$\begin{aligned}(69)\quad Ehu = \tfrac{1}{2}(1 - \nu)\rho a^2\sin\xi + (\psi' - \nu\cot\xi\,\psi)\sin\xi - \rho a^2\beta\sin^2\xi\cos\xi \\ + P_H - \nu(\sin\xi - \beta\cos\xi)P_V.\end{aligned}$$

We omit listing explicitly the corresponding expression for w if in it N_ξ and N_θ, as given by (68), are introduced.

The system (66) to (69) simplifies further, in the consideration of various subclasses of problems which may be of interest. We shall next discuss some examples of this.

11. Problems of shallow spherical shells. We now assume that the state of deformation of the shell, except for the uniform extension due to the uniform normal pressure ρ, is effectively restricted to the neighborhood of the point $\xi = 0$ such that in the region of interest

$$(70)\qquad \cot\xi \approx \frac{1}{\xi}.$$

If this is the case, it may readily be seen that some terms in equations (66) and (67) are negligible compared with others. We also assume, for simplicity's sake, that there are no horizontal loads besides the horizontal components due to the uniform normal pressure ρ and write

$$(71)\qquad P_H = 0, \qquad P_V \equiv P.$$

The simplified system of equations is then of the form

$$(72)\qquad \frac{D}{a}\left[\beta'' + \frac{\beta'}{\xi} - \frac{\beta}{\xi^2}\right] = -\Psi + \frac{\rho a^2}{2}\beta + \frac{P}{\xi} + \frac{\psi\beta}{\xi},$$

$$(73)\qquad \psi'' + \frac{\psi'}{\xi} - \frac{\psi}{\xi^2} = aC\left(\beta - \frac{\beta^2}{2\xi}\right).$$

When $P = 0$, this system of equations can be considered as forming the basis of the work of Friedrichs [5] and Tsien [11] on the nonlinear buckling theory of spherical shells. It must be noted, however, that the system (72) and (73) by itself is not sufficient to analyze the nonlinear buckling problem. Rather, appropriate energy considerations are necessary to distinguish among the solutions of (72) and (73) those which correspond to stable, indifferent, and unstable equilibrium, respectively.

When $\rho = 0$ and when the nonlinear terms are absent, the system (72) and (73) can be considered as forming the basis of earlier results on bending of shallow spherical shells [7].

A good deal of information, for the most part new, may be obtained from the two simultaneous equations for β and ψ without actually solving them, as we shall next indicate. To this end, we introduce new independent and dependent variables in accordance with the following relations:

$$\xi = \lambda\eta, \qquad \beta = \beta_0 f(\eta), \qquad \psi = \psi_0 g(\eta), \qquad P = P_0 F(\eta). \tag{74}$$

We obtain in this way

$$\frac{D\beta_0}{a\lambda^2} L(f) = -\psi_0 g + \frac{\rho a^2}{2} \beta_0 f + \frac{P_0}{\lambda}\frac{F}{\eta} + \frac{\beta_0\psi_0}{\lambda}\frac{fg}{\eta}, \tag{75}$$

$$\frac{\psi_0}{\lambda^2} L(g) = aC\beta\left(f - \frac{\beta_0}{\lambda}\frac{f^2}{2\eta}\right), \tag{76}$$

where the operator L stands for $(\ )'' + (\ )'/\eta - (\ )/\eta^2$ and where f, g, $L(f)$, $L(g)$, F/η, and f/η are all of order of magnitude unity.

12. A remark on the problem of buckling. It may be of interest to show in the following way why the nonlinear terms in equations (75) and (76) are of importance for the analysis of buckling.

We assume that bending and stretching are important, so that the highest derivatives of f and g have to be retained in the equations, and we assume that ρ satisfies the following order of magnitude relation:

$$\rho = O\left(\frac{Eh^2}{a^2}\right), \tag{77}$$

which means that the uniform normal pressure is of the same order of magnitude as the buckling pressure of the known linear theory.

The order of the system (73) and (74) is preserved if we require the following two relations to hold between the so far arbitrary parameters λ, β_0, and ψ_0:

$$\frac{D\beta_0}{a\lambda^2} = \psi_0, \qquad \frac{\psi_0}{\lambda^2} = aC\beta. \tag{78}$$

Equations (78) imply that

$$\lambda = \frac{1}{\sqrt[4]{12(1-\nu^2)}}\sqrt{\frac{h}{a}} \tag{79}$$

and

$$\frac{\beta_0}{\psi_0} = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}. \tag{80}$$

With this, equations (75) and (76) assume the form

$$f'' + \frac{f'}{\eta} - \frac{f''}{\eta^2} = -\psi + \sqrt{3(1-\nu^2)}\,\frac{pa^2}{Eh^2}f + \frac{\beta_0}{\lambda}\frac{fg}{\eta}, \tag{81}$$

$$g'' + \frac{g'}{\eta} - \frac{g''}{\eta^2} = f - \frac{\beta_0}{\lambda}\frac{f^2}{2\eta}, \tag{82}$$

The nonlinear terms in (81) and (82) will be of importance as soon as

$$\beta_0 = O(\lambda) = O\left(\sqrt{\frac{h}{a}}\right). \tag{83}$$

From equation (63), it follows that the deflection w of the shell is of the order of magnitude

$$w = O(a\beta_0\lambda). \tag{84}$$

Combination of (84), (83), and (79) shows that the nonlinear terms are of importance in the buckling problem as soon as the order of magnitude of the deflection w is that of the thickness h of the shell.

13. Shallow spherical shell with uniform radial load and edges restrained against motion. In this section we consider the solution of the differential equations (72) and (73) under the assumption that the only external load is the uniform normal pressure p and that, along the edge $\xi = \xi_e$ of the shell, we have the conditions of vanishing horizontal displacement u and bending moment M_ξ (Fig. 5). In view of (69) and (62), these conditions may be written in the form

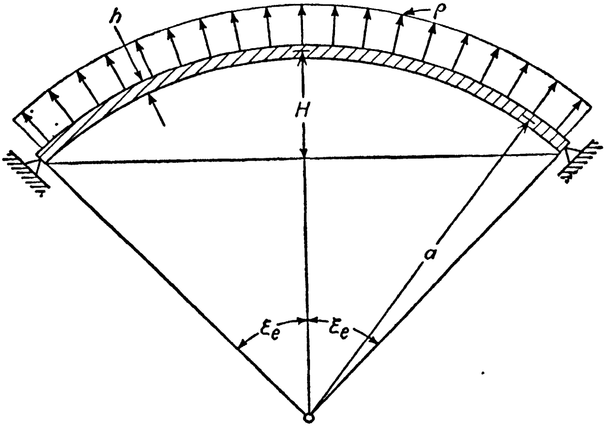

FIG. 5

$$\xi = \xi_e, \qquad \frac{d\psi}{d\xi} - \nu\frac{\psi}{\xi} = \frac{1-\nu}{2}\,pa^2, \qquad \frac{d\beta}{d\xi} + \nu\frac{\beta}{\xi} = 0. \tag{85}$$

The angular coordinate ξ_e may be expressed in terms of the height H and the radius a of the shell in the form $H = a(1 - \cos \xi_e)$ or, since $\xi_e \ll 1$,

$$\xi_e = \sqrt{\frac{2H}{a}}. \tag{86}$$

We now introduce again in equations (72) and (73) a dimensionless coordinate η and dimensionless variables f and g as in (74), which changes the differential equations into the form (75) and (76) with the simplification that now $P_0 = 0$. The boundary conditions (85) assume the form

$$\eta = \frac{\xi_e}{\lambda}, \qquad \frac{\psi_0}{\lambda}\left(\frac{dg}{d\eta} - \nu\frac{g}{\eta}\right) = \frac{1-\nu}{2}\,pa^2, \qquad \frac{df}{d\eta} + \nu\frac{f}{\eta} = 0. \tag{87}$$

We now set

$$\frac{\psi_0}{\lambda} = pa^2, \tag{88}$$

and this reduces equations (75) and (76) to the following form:

$$\frac{D\beta_0}{pa^3\lambda^3}\,L(f) = -g + \frac{\beta_0}{\lambda}\left(\frac{1}{2}f + \frac{1}{\eta}fg\right), \tag{89}$$

$$\frac{pa}{\beta_0\lambda C}\,L(g) = f - \frac{\beta_0}{\lambda}\frac{1}{2\eta}f^2. \tag{90}$$

The boundary conditions for the system (89) and (90) are, in view of (87) and (88),

$$\eta = \frac{\xi_e}{\lambda}, \qquad \frac{dg}{d\eta} - \nu\frac{g}{\eta} = \frac{1-\nu}{2}, \qquad \frac{df}{d\eta} + \nu\frac{f}{\eta} = 0. \tag{91}$$

To this is to be added the following condition for a shell without a central hole:

$$\eta = 0; \qquad f, g \text{ regular}. \tag{92}$$

The system (89) and (90) contains three parameters

$$k_1 = \frac{D\beta_0}{pa^3\lambda^3}, \qquad k_2 = \frac{pa}{\beta_0\lambda C}, \qquad k_3 = \frac{\beta_0}{\lambda}, \tag{93}$$

and in these the two quantities β_0 and λ may be assigned at will. We next show that appropriate disposition of β_0 and λ permits a classification of the present shell problem which at once determines the range of applicability of previously known methods of analysis and which furnishes a simplified system of equations for a transition range in which neither of the previously known methods applies.[2] We will further show that for each of the three classes of problems the order of magnitude of the angle ξ_e determines whether or not we are dealing with a boundary-layer phenomenon.

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[2] For an earlier application of this kind of reasoning to a problem in aerodynamics, see the paper *On two-dimensional non-steady motion of a slender body in a compressible fluid* by

We subdivide the problem into three main classes in the following manner:

$$\begin{array}{llll} \text{I.} & k_1 = 1, & k_2 = 1, & k_3 = 0(1); \\ \text{II.} & k_1 = 0(1), & k_2 = 1, & k_3 = 1; \\ \text{III.} & k_1 = 1, & k_2 = 0(1), & k_3 = 1. \end{array} \tag{94}$$

Equations (94) imply the following values of β_0, λ, and of the disposable parameter k_n.

$$\begin{array}{llll} \text{I.} & \lambda = \dfrac{1}{m}\sqrt{\dfrac{h}{a}}, & \beta_0 = m\,\dfrac{\rho a^2}{Eh^2}\sqrt{\dfrac{h}{a}}, & k_3 = m^2\dfrac{\rho a^2}{Eh^2}; \\ \text{II.} & \lambda = \beta_0 = \sqrt{\dfrac{\rho a}{Eh}}, & k_1 = \dfrac{1}{m^4}\left(\dfrac{Eh^2}{\rho a^2}\right)^2 & \\ \text{III.} & \lambda = \beta_0 = \dfrac{1}{m^2}\sqrt{\dfrac{Eh^3}{\rho a^3}}, & k_2 = m^4\left(\dfrac{Eh^2}{\rho a^2}\right)^2, & \end{array} \tag{95}$$

where the quantity m is given by

$$m = \sqrt[4]{12(1 - \nu^2)}. \tag{96}$$

We find that class III is either contained in classes I and II or is leading to contradictions. Consequently, attention may be restricted to classes I and II.

For class I we have the following boundary-value problem:

$$\begin{aligned} L(f) &= -g + m^2\,\frac{\rho a^2}{Eh^2}\left(\frac{1}{2}f + \frac{1}{\eta}fg\right), \\ L(g) &= f - m^2\,\frac{\rho a^2}{Eh^2}\left(\frac{1}{2\eta}f^2\right), \end{aligned} \tag{97}$$

$$\eta = m\sqrt{\frac{2H}{h}}, \qquad \frac{dg}{d\eta} - \nu\frac{g}{\eta} = \frac{1-\nu}{2}, \qquad \frac{df}{d\eta} + \nu\frac{f}{\eta} = 0. \tag{98}$$

For class II the boundary-value problem assumes the form

$$\begin{aligned} \frac{1}{m^4}\left(\frac{Eh^2}{\rho a^2}\right)^2 L(f) &= -g + \frac{1}{2}f + \frac{1}{\eta}fg, \\ L(g) &= f - \frac{1}{2\eta}f^2, \end{aligned} \tag{99}$$

$$\eta = \sqrt{\frac{2H/h}{\rho a^2/Eh^2}}, \qquad \frac{dg}{d\eta} - \nu\frac{g}{\eta} = \frac{1-\nu}{2}, \qquad \frac{df}{d\eta} + \nu\frac{f}{\eta} = 0. \tag{100}$$

C. C. Lin, E. Reissner, and H. S. Tsien, Journal of Mathematics and Physics vol. 27 (1948) pp. 220–231.

Class I contains a subclass defined by the order of magnitude relation

$$\frac{\rho a^2}{Eh^2} \ll 1 \tag{101}$$

For this subclass, the terms having the factor $\rho a^2/Eh^2$ on the right of equations (98) may be neglected, and we obtain the equations which follow from the linearized bending theory as first developed by H. Reissner [10].

So far nothing has been said about the existence of the edge effect in this theory. The edge effect will occur when the dimensionless edge coordinate $\eta = \eta_e$ is large compared with unity, that is, when

$$1 \ll \frac{H}{h}. \tag{102}$$

In this instance, the terms with $1/\eta$ in (97) and (98) may be neglected, and we have, instead of (97) and (98), the following system:

$$\begin{gathered} f'' = -g + m^2 \frac{\rho a^2}{Eh^2} \frac{f}{2}, \qquad g'' = f, \\ \eta = m\sqrt{\frac{2H}{h}}, \qquad \frac{dg}{d\eta} = \frac{1-\nu}{2}, \qquad \frac{df}{d\eta} = 0. \end{gathered} \tag{103}$$

We thus find that, when $\rho a^2/Eh^2 = O(1)$, the condition (102) for the existence of the edge effect permits linearization of the differential equations of the problem even for values of $\rho a^2/Eh^2$ which place the problem outside the range of the linearized bending theory.

It remains to discuss the class of problems for which

$$1 \ll \frac{\rho a^2}{Eh^2}. \tag{104}$$

We find that equations (99) and (100) of class II, under the condition (104), reduce to the following:

$$\begin{gathered} 0 = -g + \frac{f}{2} + \frac{fg}{\eta}, \qquad L(g) = f - \frac{f^2}{2\eta}, \\ \eta = \sqrt{\frac{2H/h}{\rho a^2/Eh^2}}, \qquad \frac{dg}{d\eta} - \nu\frac{g}{\eta} = \frac{1-\nu}{2}. \end{gathered} \tag{105}$$

Together with the derivatives of f in the differential equations, we must disregard the boundary conditions on f.

An edge effect will occur whenever

$$\frac{\rho a^2}{Eh^2} \ll \frac{H}{h}, \tag{106}$$

and then all terms which contain η explicitly may again be disregarded. This reduces (105) to the following linear system

$$-g + \tfrac{1}{2}f = 0, \qquad g'' = f, \tag{107}$$
$$\eta = \sqrt{\frac{2H/h}{\rho a^2/Eh^2}}, \qquad \frac{dg}{d\eta} = \frac{1-\nu}{2}.$$

Equations (107) may be considered to be the basis of the boundary-layer solution of nonlinear sheet theory which was discovered by Bromberg and Stoker [2]. Our contribution to this solution is recognition of the fact that its range of applicability is limited by two explicit order-of-magnitude relations, namely, equation (104) which may be interpreted as a condition concerning permissible sheet thicknesses ($h \ll a\sqrt{\rho/E}$), and equation (106) which limits the degree of shallowness of the shell which is permitted.

One other question is of interest concerning this boundary-layer solution of nonlinear sheet theory and that is the order of magnitude of the bending stresses which it implies and which are neglected. We have

$$\sigma_B = O\left(\frac{Eh}{a}\frac{d\beta}{d\xi}\right) = O\left(\frac{Eh}{a}\frac{\beta_0}{\lambda}\frac{df}{d\eta}\right) = O\left(\frac{Eh}{a}\right).$$

This must be compared with the direct stresses σ_D which are $O(\rho a/h)$. Thus $\sigma_B/\sigma_D = O(Eh^2/\rho a^2)$, and consequently we have $\sigma_B \ll \sigma_D$, which is as it should be.

We may summarize the results of this section as follows:

1. Linear bending theory applies when $\rho a^2/Eh^2 \ll 1$.
2. Both bending and membrane action must be taken into account when $\rho a^2/Eh^2 = O(1)$. In this range, the equations of the problem are linear as long as $h \ll H$; otherwise the equations are nonlinear.
3. Bending action can be disregarded when $1 \ll \rho a^2/Eh^2$. In this range, the equations of the problem are linear as long as $h \ll HEh^2/\rho a^2$; otherwise they are nonlinear.

In the foregoing we have assumed that no buckling phenomenon needs to be considered. As it is known that buckling occurs when $|\rho a^2/Eh^2| = O(1)$ and ρ is negative, it follows that our conclusions apply regardless of the direction of the pressure ρ only in the range of applicability of the linear bending theory. In the remaining range, ρ must be positive, *i.e.*, the pressure must be directed outward.

14. Remarks on the extension of the foregoing results to nonshallow shells and to other load distributions. While the preceding discussion was for a shallow shell, characterized by the assumption that $\cot \xi \doteq 1/\xi$, it may be carried over directly into the range $\xi = O(1)$ as long as the deviations from the condition of uniform radial expansion are restricted to narrow edge zones. We omit the details and merely state that, in this case, equations (66) and (77) reduce to the following:[3]

$$D\beta'' + a(\psi - \tfrac{1}{2}\rho a^2\beta) = 0, \qquad \psi'' - aC\beta = 0, \tag{108}$$

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[3] A better approximation, which in the linear bending case gives the first term of a true asymptotic development, is obtained if, before simplification, first-derivative terms are eliminated in (III) and (IV).

with the boundary conditions

$$\psi'(\xi_e) = -\frac{1-\nu}{2} pa^2, \qquad \beta'(\xi_e) = 0. \tag{109}$$

When $pa^2 \ll Eh^2$, the term with p in (108) may be omitted. When $Eh^2 \ll pa^2$, the term with D in (108) may be omitted together with the boundary condition for β. In the in-between range, the complete system (108) and (109) should be used. The differential equations of the problem (but not the problem itself) are seen to be linear in the entire range of values of the parameter pa^2/Eh^2. Applicability (108) and (109) presupposes again that $pa \ll Eh$, which condition is equivalent to the condition that the strain associated with uniform radial expansion due to a pressure p is a small strain.

It suggests itself to propose the following generalization of this procedure: Given any shell of revolution with any kind of load distribution such that, apart from the edge effect, the linear membrane solution applies for sufficiently low load intensity. As the load intensity is increased, it may or may not, depending on its direction, cause buckling. In the latter case, take account of it in the analysis of the edge effect, roughly as follows: Assume in equations (III) and (IV) that

$$\Psi = \Psi_m + \psi, \tag{110}$$

where Ψ_m is the result of linear membrane theory, *i.e.*,

$$\Psi_m \sin\phi_0 = (r_0 V)\cos\phi_0, \tag{111}$$

and where ψ is due to edge restraint other than that of the linear membrane theory solution.

Introduce (110) into the last term of equation (III) and neglect terms to the extent that this equation assumes the following form:

$$\beta'' = -\frac{\alpha_0^2}{r_0 D}\left(\psi \sin\phi_0 - \frac{\Psi_m}{\cos\phi_0}\beta\right). \tag{112}$$

Corresponding to (112), obtain from equation (IV) the following simplified equation:

$$\psi'' = \frac{\alpha_0^2 C}{r_0}\beta \sin\phi_0. \tag{113}$$

It is believed that this system of equations will take account, in first approximation, of the change in the nature of the edge effect which takes place as the intensity of a given loading is made higher and higher.

In the limit of very high-load intensity, we may again have a nonlinear membrane theory such that equation (112) is simplified to

$$\beta = \frac{\psi}{\Psi_m}\sin\phi_0\cos\phi_0, \tag{114}$$

and the differential equation for ψ becomes

$$\psi'' = \left(\frac{\alpha_0^2 C}{r_0} \frac{\sin^2 \phi_0 \cos \phi_0}{\Psi_m}\right) \psi. \tag{115}$$

For the spherical shell with uniform radial pressure ρ, equation (115) reduces, as it should, to the form

$$\psi'' = 2 \frac{Eh}{\rho a} \psi, \tag{116}$$

which corresponds to the result of Bromberg and Stoker [2].

15. Prestressed shallow spherical shell with point load at the apex. We propose now to discuss two questions: (1) the transition from linear theory to nonlinear theory for the problem of the spherical shell acted upon by a point load P_a at the apex of the shell and (2) the effect of a uniform radial pressure ρ on stresses and reflections due to the point load P_a. Since for a point load P_a we have that $2\pi r_0 V = P_a$, it follows, in view of (64), that the function $P_v = P$ in equation (72) is simply $P = P_a/2\pi$, and then $P_0 F$ in (75) as defined by (74) is given by $P_a/2\pi$. We may then write equations (75) and (76) in the following form:

$$\begin{aligned} k_3 L(f) &= -k_4 g \left(1 - \frac{k_2}{\eta} f\right) + \frac{1}{\eta} + k_5 f, \\ k_1 L(g) &= f\left(1 - \frac{k_2}{2\eta} f\right), \end{aligned} \tag{117}$$

where the dimensionless parameters k_n are given by

$$\begin{aligned} k_1 &= \frac{\psi_0}{aC\beta_0\lambda^2}, \qquad k_2 = \frac{\beta_0}{\lambda}, \qquad k_3 = \frac{2\pi D\beta_0}{aP_a\lambda}, \\ k_4 &= \frac{2\pi\psi_0\lambda}{P_a}, \qquad k_5 = \frac{\pi\rho a^2\beta_0}{P_a}. \end{aligned} \tag{118}$$

In these five parameters we have at our disposal the three quantities ψ_0, β_0, and λ. This means that we may set three of the k_n equal to unity. Whenever one of the remaining parameters is small compared with unity, we may omit the term multiplied by it in the differential equations (117). In this way, we may classify various types of theories for the present problem and establish the range of validity of these theories. A linearized system of equations is applicable whenever we have $k_2 \ll 1$.

We shall carry through this discussion for the case

$$k_5 = 0, \tag{119}$$

which corresponds to a shell without radial preload. The general case may be treated analogously.

When $k_5 = 0$, we have the following four possibilities:

$$\begin{array}{llll} \text{(i)} & k_1 = k_2 = k_3 = 1, & \text{(ii)} & k_1 = k_2 = k_4 = 1, \\ \text{(iii)} & k_1 = k_3 = k_4 = 1, & \text{(iv)} & k_2 = k_3 = k_4 = 1. \end{array} \tag{120}$$

It is found that (i) and (iv) leave λ undetermined and thus are not usable. Thus (ii) and (iii) should cover the entire range of possibilities and this indeed is found to be the case, which may be seen as follows:

(ii) From

$$\frac{\psi_0}{aC\beta_0\lambda^2} = 1, \qquad \frac{\beta_0}{\lambda} = 1, \qquad \frac{2\pi\psi_0\lambda}{P_a} = 1, \tag{121}$$

there is obtained

$$\lambda = \beta_0 = \sqrt[4]{\frac{P_a}{2\pi ahE}}, \qquad \psi_0 = \frac{P_a}{2\pi\lambda}, \tag{122}$$

and

$$k_3 = \frac{2\pi D}{aP_a} = \frac{\pi}{6(1-\nu^2)}\,\frac{Eh^2}{P_a}\,\frac{h}{a}. \tag{123}$$

The differential equations (117) are thus

$$\frac{2\pi D}{aP_a}L(f) = -g\left(1 - \frac{f}{\eta}\right) + \frac{1}{\eta}, \qquad L(g) = f\left(1 - \frac{f}{2\eta}\right). \tag{124}$$

When $2\pi D/aP_a = O(1)$, all terms in equation (124) must be retained. When $2\pi D/aP_a \ll 1$, the term $L(f)$ in (124) may be omitted, and we have a nonlinear membrane theory. The case $2\pi D/aP_a \gg 1$ does not fall within the scope of (124).

On the basis of (124) we may write, without solving the differential equations,

$$\begin{aligned} \beta &= \sqrt[4]{\frac{P_a/2\pi}{aEh}}\, f\left(\frac{\xi}{\sqrt[4]{\dfrac{P_a/2\pi}{ahE}}}, \frac{D}{aP_a}\right), \\ \psi &= \frac{P_a/2\pi}{\sqrt[4]{\dfrac{P_a/2\pi}{ahE}}}\, g\left(\frac{\xi}{\sqrt[4]{\dfrac{P_a/2\pi}{ahE}}}, \frac{D}{aP_a}\right), \end{aligned} \tag{125}$$

and the assumption $D/aP_a = 0$ gives the results of a nonlinear membrane theory. We have in the latter case, from (124),

$$f = \eta - \frac{1}{g}, \tag{126a}$$

and therewith the following nonlinear differential equation for g:

$$L(g) = \frac{\eta}{2}\left[1 - \frac{1}{\eta^2 g^2}\right]. \tag{126b}$$

Further consideration will be needed to see to what extent such a nonlinear membrane theory for a shell with point load can be of practical significance.

We now return to equations (117) to (120) to consider the remaining case (iii).

(iii). From

$$\frac{\psi_0}{aC\beta_0\lambda^2} = 1, \qquad \frac{2\pi D\beta_0}{a\lambda P_a} = 1, \qquad \frac{2\pi\psi_0\lambda}{P_a} = 1, \tag{127}$$

there is obtained

$$\lambda = \frac{1}{\sqrt[4]{12(1-\nu^2)}}\sqrt{\frac{h}{a}}, \qquad \beta_0 = \frac{aP_a}{2\pi D}\lambda, \qquad \psi_0 = \frac{P_a}{2\pi\lambda}, \tag{128}$$

and

$$k_2 = \frac{\beta_0}{\lambda} = \frac{aP_a}{2\pi D}. \tag{129}$$

The differential equations (117) are now of the form

$$\begin{aligned} L(f) &= -\left(1 - \frac{aP_a}{2\pi D}\frac{f}{\eta}\right)g + \frac{1}{\eta}, \\ L(g) &= \left(1 - \frac{aP_a}{2\pi D}\frac{f}{\eta}\right)f. \end{aligned} \tag{130}$$

When $aP_a/2\pi D = O(1)$, the nonlinear terms must be retained. When $aP_a/2\pi D \ll 1$, the nonlinear terms may be neglected. As cases (ii) and (iii) together cover continuously the range of all possible values of aP_a/D, it follows that cases (i) and (iv) at best could have done no more than duplicate the information obtainable from cases (ii) and (iii).

Corresponding to equations (125), we now have the following representation for β and ψ:

$$\begin{aligned} \beta &= \frac{P_a a}{2\pi D}\frac{\sqrt{h/a}}{\sqrt[4]{12(1-\nu^2)}} f\left(\frac{\sqrt[4]{12(1-\nu^2)}}{\sqrt{h/a}}\,\xi, \frac{P_a a}{D}\right), \\ \psi &= \frac{P_a}{2\pi}\sqrt[4]{12(1-\nu^2)}\,g\left(\frac{\sqrt[4]{12(1-\nu^2)}}{\sqrt{h/a}}\,\xi, \frac{P_a a}{D}\right). \end{aligned} \tag{131}$$

The formulas of the linearized theory of bending are obtained from (130) and (131) by setting in them $aP_a/D = 0$, with the exception of the first factor in β which remains aP_a/D.

Let us finally discuss more fully the condition for linearity which may be written in the form

$$P_a \ll \frac{\pi}{6(1-\nu^2)}\frac{Eh^3}{a}. \tag{132}$$

In order to express this result in terms of permissible displacements, we use a previously derived result for the center deflection w_a according to the linearized theory of bending [7]. We have $w_a = \frac{1}{4}\sqrt{3(1-\nu^2)}\,(P_a a/Eh^2)$. Accordingly, the condition (132) may be written in the form

$$w_a \ll \frac{\pi\sqrt{3}}{24(1-\nu^2)}\,h. \tag{133}$$

Thus we have the result that, for this shell problem, nonlinear effects appear as soon as the deflection is of the same order of magnitude as the wall thickness of the shell. This is in contrast with some other shell problems for which a much wider range of linearity exists, as discussed earlier in this paper.

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ON ASYMPTOTIC INTEGRATION IN SHELL THEORY[1]

BY

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1. Introduction. When the deformations due to transverse shear and normal stresses are neglected, the analysis of rotationally symmetric deformations of thin elastic shells of revolution can be made to depend upon the solution of two simultaneous ordinary differential equations of the second order. The case considered here is that in which the deformations are small, so that the governing equations are linear.

The problem was formulated in this manner first for the spherical shell of constant thickness by H. Reissner [1], and afterward for the general shell of revolution by E. Meissner [2, 3]. H. Reissner also discovered that, to a good approximation, the complete solution could be expressed as the sum of two parts, one part being the so-called membrane solution for distributed surface loads, and the other part being a state of bending due to edge forces and moments which modifies the membrane solution only in relatively narrow edge zones. An explicit proof of this assertion was given [1] for the case of the spherical shell subject only to its own weight. The possibility of expressing the localized effects in terms of asymptotic developments, in powers of the ratio of the shell thickness to a representative dimension of the shell, was also suggested by him.

O. Blumenthal [4] obtained such developments, in the case of the spherical shell of constant thickness, by a method which is particularly well adapted to the determination of bounds on the error resulting from terminating the series at a given stage, and E. Steuermann [5] used the same method in indicating the derivation of the leading term in each such development in the more general case. Numerous applications of these methods have been given.

In the present paper, the asymptotic developments are obtained for the general shell of revolution, and the results are presented explicitly in a form suitable for direct application. Further, an investigation is made of the validity of using the so-called membrane solution in the general case as an approximation to an inhomogeneous solution of the governing differential equations, corresponding to the presence of distributed surface loading.

2. Notations and conventions. The physical quantities of principal interest are represented in Fig. 1, which shows half of an axial section of the shell. In addition to the indicated meridional direct stress resultant N_ξ, the transverse shear resultant Q, and the bending moment M_ξ, there exist also the circumferential direct stress resultant N_θ and the bending moment M_θ, acting on a section θ = constant. The radial ("horizontal") resultant H and the axial ("vertical") resultant V are also indicated.

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The radial and axial coordinates of a point on the middle surface of the shell are r and z, the slope of a meridian ϕ, the shell thickness h, and position along a meridian is denoted by the coordinate ξ. Of the two radii of principal normal curvature of the middle surface, the first, which is numerically equal to the distance along the normal from the middle surface to the axis of revolution, is denoted by R, while the radius of curvature of the meridian is denoted by R_1.

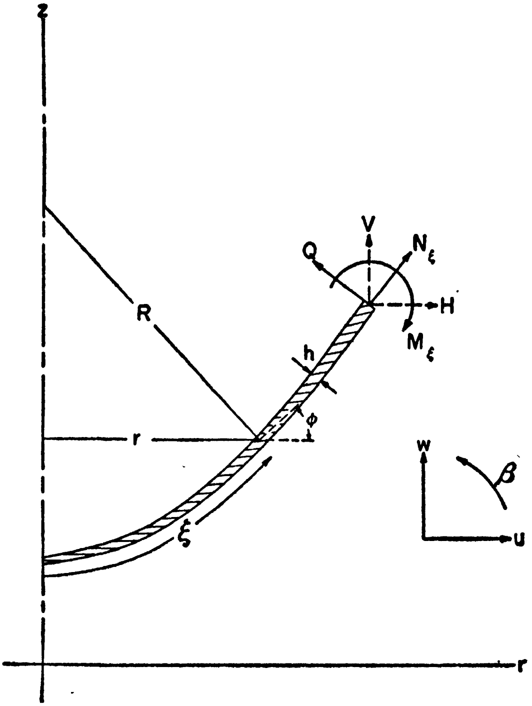

FIG. 1

The parameter α is defined by the relation

$$ds = \alpha\, d\xi, \tag{1}$$

where s is arc length along the meridian. It is assumed that the definition of ξ is such that α is of the order of magnitude of a representative dimension of the shell.

Attention is restricted to those cases in which the meridian has a continuously turning tangent which is not perpendicular to the axis of symmetry except perhaps at an apex on that axis. Thus, for example, the treatment of complete toroidal shells is not included in the present work.

A prime is to be used to indicate differentiation with respect to ξ.

From the differential geometry of the shell, one has the relation

$$z' = \frac{\alpha r}{R}, \tag{2}$$

if the sign of R is suitably chosen. The direction of the z axis is here to be chosen in such a way that z', and hence also R, are nonnegative. The relations

$$\phi' = \frac{\alpha}{R_1}, \qquad r' = \alpha \cos \phi \tag{3a, b}$$

and

$$r = R \sin \phi \tag{4}$$

will also be used.

3. Formulation of the problem. As is well known, all the quantities specifying the state of deformation of the shell can be conveniently expressed as functions of the prescribed loading in terms of the rotation β of the meridian and a parameter U defined as the product of the radius of curvature R and the shear resultant Q,

$$U = RQ. \tag{5}$$

The relations involved can be summarized in the form

$$rH = -U + RV \cos \phi, \tag{6}$$

$$rN_\xi = -U \cos \phi + RV, \qquad N_\theta = -\frac{1}{\alpha} U' - \frac{R^2}{rR_1} V + Rp_n, \tag{7a, b}$$

$$\alpha M_\xi = -D\left(\beta' + \nu \frac{r'}{r} \beta\right), \qquad \alpha M_\theta = -D\left(\frac{r'}{r} \beta + \nu\beta'\right), \tag{8a, b}$$

$$Ehu = r(N_\theta - \nu N_\xi), \qquad w = \int \left(\beta + \frac{r}{R\alpha} u'\right) \frac{\alpha^2}{r'} \, d\xi, \tag{9a, b}$$

where the vertical resultant V is given by

$$rV = -\int \alpha r p_z \, d\xi. \tag{10}$$

In these equations, p_z represents the distributed vertical load intensity, positive in the positive z direction, and p_n is the normal load intensity, positive outward. The bending stiffness D is defined by the relation

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \tag{11}$$

where E is Young's modulus and ν is Poisson's ratio.

The quantities β and U, to be determined, then satisfy the differential equations

$$L_1\beta + \frac{\alpha^2}{DR}U = 0, \tag{12a}$$

$$L_2U - \frac{Eh\alpha^2}{R}\beta = -\frac{Eh\alpha^2}{R}\beta_0, \tag{12b}$$

where L_1 and L_2 are the linear differential operators

$$L_1 = \frac{d^2}{d\xi^2} + \frac{(rh^3/\alpha)'}{rh^3/\alpha}\frac{d}{d\xi} - \left(\frac{r'}{r}\right)^2 + \nu\frac{(r'h^3/\alpha)'}{rh^3/\alpha}, \tag{13a}$$

$$L_2 = \frac{d^2}{d\xi^2} + \frac{(r/\alpha h)'}{r/\alpha h}\frac{d}{d\xi} - \left(\frac{r'}{r}\right)^2 - \nu\frac{(r'/\alpha h)'}{r/\alpha h}, \tag{13b}$$

and β_0 is a function of the prescribed loading which can be expressed in the form

$$\begin{aligned}\beta_0 = \frac{R}{Eh}\Bigg\{&\left[\left(1 - \frac{R^2}{R_1^2}\right)\cot\phi + \frac{R}{\alpha}\left(\frac{R}{R_1}\right)' - \frac{Rh'}{\alpha h}\left(\nu + \frac{R}{R_1}\right)\right]\frac{V}{r} \\ &- \left(2 - \frac{R}{R_1} + \nu\right)p_n\cot\phi - \frac{Rh}{\alpha}\left(\frac{p_n}{h}\right)' - \left(\nu + \frac{R}{R_1}\right)p_z\csc\phi\Bigg\}.\end{aligned} \tag{14}$$

For present purposes, it is desirable to transform these equations in such a way that first derivatives of the dependent variables are not present. This transformation can be accomplished in a symmetrical way by the introduction of the new dependent variables X and Y defined by the relations

$$\beta = \sqrt{\frac{\alpha}{RD}}X, \qquad U = \sqrt{\frac{Eh\alpha}{r}}Y, \tag{15}$$

according to which equations (12a, b) can be put into the form

$$X'' - \theta_1X + 2k^2\Phi^2Y = 0, \tag{16a}$$

$$Y'' - \theta_2Y - 2k^2\Phi^2X = -2k^2\Phi^2X_0, \tag{16b}$$

where the coefficients θ_1 and θ_2 are given by

$$\theta_1 = \left(\frac{r'}{r}\right)^2 - \nu\frac{(r'h^3/\alpha)'}{rh^3/\alpha} + \frac{1}{2}\left[\frac{(rh^3/\alpha)'}{rh^3/\alpha}\right]' + \frac{1}{4}\left[\frac{(rh^3/\alpha)'}{rh^3/\alpha}\right]^2, \tag{17a}$$

$$\theta_2 = \left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/\alpha h)'}{r/\alpha h} + \frac{1}{2}\left[\frac{(r/\alpha h)'}{r/\alpha h}\right]' + \frac{1}{4}\left[\frac{(r/\alpha h)'}{r/\alpha h}\right]^2. \tag{17b}$$

In these equations, the constant parameter k is defined by the relation

$$k^2 = \sqrt{3(1 - \nu^2)}\left(\frac{\alpha^2}{Rh}\right)_{\mathrm{ref}} \tag{18}$$

and the function Φ is given by

$$\Phi(\xi) = \sqrt{\frac{\alpha^2/Rh}{(\alpha^2/Rh)_{\mathrm{ref}}}} \tag{19}$$

where a representative reference value of the ratio α^2/Rh is to be taken in (18). The auxiliary function X_0 is given by

$$X_0 = \sqrt{\frac{RD}{\alpha}}\,\beta_0. \tag{20}$$

For most thin shells considered in practice, the constant k is large with respect to unity. In particular, for a spherical or cylindrical shell, one may identify R and α with the radius a of the middle surface, and k is then proportional to the ratio $\sqrt{a/h}$. In what follows, it is assumed that the thickness h and the radii R and R_1 may vary appreciably only over a distance comparable with a representative dimension of the shell. Specifically, it is postulated that the ξ derivative of a parameter specifying the shape of the shell shall not be large with respect to that parameter. The coefficients θ_1, θ_2, and Φ in (16) are then of order of magnitude unity, and the method of asymptotic integration under consideration is appropriate.

It is convenient to consider separately the homogeneous and inhomogeneous solutions of (16*a*, *b*), corresponding, respectively, to the absence and presence of distributed surface loading, and to obtain the complete solution by superposition.

4. Homogeneous solutions by asymptotic integration. As a first step toward solving the homogeneous equations corresponding to (16*a*, *b*), it is supposed that homogeneous solutions exist in the form

$$X_H = xe^{k\zeta}, \qquad Y_H = ye^{k\zeta}, \tag{21}$$

where the unknown functions, x, y, and ζ tend to finite nonzero limits as $k \to \infty$. If these assumptions are introduced into (16*a*, *b*) with $X_0 = 0$, the requirement that the resultant equations be compatible for large values of k determines permissible expressions for ζ in the form

$$\zeta = \omega \int \Phi \, d\xi, \tag{22}$$

where ω satisfies the equation $\omega^4 + 4 = 0$. As will be seen, attention may be restricted to the two values of ω for which

$$\omega^2 = 2i, \tag{23}$$

without loss of generality.

Corresponding to either determination of ζ, the functions x and y are then formally assumed as series of inverse powers of k, with coefficients to be determined as functions of ξ. By introducing the expressions

$$X_H = e^{k\omega\int\Phi\, d\xi} \sum_{n=0}^{\infty} x_n(\xi)k^{-n}, \qquad Y_H = e^{k\omega\int\Phi\, d\xi} \sum_{n=0}^{\infty} y_n(\xi)k^{-n}, \tag{24}$$

into the left-hand members of (16*a*, *b*), and formally equating to zero the resultant coefficients of successive powers of k, the two recurrence formulas

$$2i\Phi^2(x_n - iy_n) + 2\omega\sqrt{\Phi}(\sqrt{\Phi}x_{n-1})' + (x''_{n-2} - \theta_1 x_{n-2}) = 0, \tag{25a}$$

$$2\Phi^2(x_n - iy_n) - 2\omega\sqrt{\Phi}(\sqrt{\Phi}y_{n-1})' - (y''_{n-2} - \theta_2 y_{n-2}) = 0, \tag{25b}$$

are obtained. These conditions hold for $n = 0, 1, 2, \cdots$, with the convention that a quantity with a negative subscript is to be taken as zero.

A more convenient formulation is obtained by introducing the abbreviations

$$z_n = x_n + iy_n, \qquad \bar{z}_n = x_n - iy_n. \tag{26}$$

It should be noticed, however, that since x_n and y_n are generally complex quantities, the functions z and $\bar{z}$ are generally not conjugate. If two new equivalent equations are obtained by forming the sum and difference of (25*a*, *b*), the resultant equations can be written in the form

$$x_n - iy_n = \frac{i\omega}{2\Phi}\left(\bar{z}_{n-1} + \frac{\Phi'}{2\Phi}\bar{z}'_{n-1}\right) + \frac{i}{4\Phi^2}(\bar{z}''_{n-2} - f\bar{z}_{n-2} - gz_{n-2}), \tag{27a}$$

$$x_n + iy_n = -\frac{1}{2\omega\sqrt{\Phi}}\int (z''_{n-1} - fz_{n-1} - g\bar{z}_{n-1})\frac{d\xi}{\sqrt{\Phi}} + \frac{c_n}{\sqrt{\Phi}}, \tag{27b}$$

where

$$f = \tfrac{1}{2}(\theta_1 + \theta_2), \qquad g = \tfrac{1}{2}(\theta_1 - \theta_2). \tag{28a, b}$$

The constant c_n in (27*b*) is an arbitrary constant of integration.

With $n = 0$, equations (27*a*, *b*) give $x_0 - iy_0 = 0$ and $x_0 + iy_0 = c_0/\sqrt{\Phi}$, and hence, with the convenient choice $c_0 = 2$, the functions x_0 and y_0 are determined in the form

$$x_0 = \frac{1}{\sqrt{\Phi}}, \qquad y_0 = -ix_0. \tag{29}$$

Next, with $z_0 = 2x_0$ and $\bar{z}_0 = 0$, there follows

$$x_1 = -\frac{x_0}{2\omega}\int (x_0'' - fx_0)x_0\, d\xi + c_1 x_0, \qquad y_1 = -ix_1, \tag{30}$$

and the succeeding coefficient functions in (24) may be determined in a similar way.

Thus, for each value of ω,

$$\omega = \pm(1 + i), \tag{31}$$

asymptotic homogeneous solutions of (16) are formally determined by (24). Since the coefficients in (16) are real, it follows that the real or imaginary parts of the complex solutions so obtained are also solutions.

5. Example. Spherical shell of constant thickness. To illustrate these developments, the case of a spherical shell of constant thickness ($h' = 0$) may be considered. With the relations

$$\xi = \phi, \qquad \alpha = R = a, \qquad r = a \sin \phi,$$

there follows, from (17),

$$\theta_1 = \tfrac{1}{4}(3 \csc^2 \phi - 5 + 4\nu), \qquad \theta_2 = \tfrac{1}{4}(3 \csc^2 \phi - 5 - 4\nu),$$

and hence, from (28),

$$f = \tfrac{1}{4}(3 \csc^2 \phi - 5), \qquad g = \nu.$$

Also, equations (18) and (19) give the definitions

$$k^2 = \sqrt{3(1 - \nu^2)}\, \frac{a}{h}, \qquad \Phi = 1.$$

From equations (29) and (30), there follows

$$x_0 = 1, \qquad y_0 = -i,$$

$$x_1 = -\frac{1}{8\omega}(3 \cot \phi + 5\phi + c_1), \qquad y_1 = -ix_1,$$

where c_1 is an arbitrary constant. Hence, equations (24) and (15) give the asymptotic expansions

$$\beta_H = \frac{e^{\omega(k\phi+A)}}{\sqrt{D}\sin\phi}\left[1 - \frac{3\cot\phi + 5\,\phi + c_1}{8\omega k} + O(k^{-2})\right],$$

$$aQ_H = -i\frac{e^{\omega(k\phi+A)}}{\sqrt{\dfrac{\sin\phi}{Eh}}}\left[1 - \frac{3\cot\phi + 5\phi + c_1}{8\omega k} + O(k^{-2})\right],$$

where A is an arbitrary complex constant and $\omega = \pm(1 + i)$.

With the more convenient notation

$$e^{\omega A} = C\sqrt{D}e^{\pm ic},$$

where C and c are real arbitrary constants, the results of taking real parts of these developments are expressions of the form

$$\beta_H = \frac{Ce^{\pm k\phi}}{\sqrt{\sin\phi}}\left[\cos(k\phi + c) \mp \frac{3\cot\phi + 5\phi + c_1}{8\sqrt{2}k}\cos\left(k\phi + c \mp \frac{\pi}{4}\right) + O(k^{-2})\right],$$

$$Q_H = \pm\frac{Eh^2}{\sqrt{12(1-\nu^2)}a}Ce^{\pm k\phi}\left[\sin(k\phi + c)\right.$$
$$\left.\mp \frac{3\cot\phi + 5\phi + c_1}{8\sqrt{2}k}\sin\left(k\phi + c \mp \frac{\pi}{4}\right) + O(k^{-2})\right].$$

The character of the arbitrary constant c_1 may be seen by noticing that the expressions are unchanged if c_1 is replaced by zero, and C is replaced by

$C(1 \mp c_1/8\sqrt{2}k)$. Thus c_1 can be chosen arbitrarily without actually specializing the solution. However, the choice of c_1 would be reflected also in the form of terms of higher order in k^{-1}. For each choice of the ambiguous sign in the above expressions, the constants C and c are arbitrary. Thus, four independent solutions have indeed been obtained, in accordance with the fact that in general four boundary conditions are to be satisfied. In the case when the shell is closed at its apex $\phi = 0$, it can be shown that only the solutions for which the upper sign is chosen are developments of solutions which are finite at the apex. The results obtained are in agreement with known results if one sets $c_1 = 0$[4].

The exact solutions in the present case are expressible in terms of associated Legendre functions of complex order, as was first noted by A. E. H. Love. By making use of known properties of these functions, it can be shown that the developments obtained are truly asymptotic representations of solutions in any closed region which excludes the apex $\phi = 0$, in the sense that the error associated with using the first n terms of the series in place of the corresponding exact solution tends to zero, uniformly in ϕ, as k tends to infinity.

By making use of equations (6) to (9), with the loading terms suppressed, corresponding developments for the remaining quantities of interest may be obtained. For sufficiently thin shells, retention of only the leading term in each expansion provides sufficient accuracy in practical applications.

In the following section, the leading terms are listed explicitly in the more general case under consideration.

6. Leading term in the homogeneous solutions. By proceeding as in the above special case, and introducing the convenient abbreviation

$$m = \sqrt[4]{3(1 - \nu^2)}, \tag{32}$$

the real part of the leading terms in the homogeneous developments for β and RQ are obtained in the more general case, in the form

$$\beta_H = C \sqrt[4]{\frac{R}{r^2h^5}}\, e^{\pm m \int \frac{\alpha\, d\xi}{\sqrt{Rh}}} \left[\cos\left(m \int \frac{\alpha\, d\xi}{\sqrt{Rh}} + c\right) + O(k^{-1})\right],$$

$$RQ_H = \pm \frac{Eh^2}{2m^2} C \sqrt[4]{\frac{R}{r^2h^5}}\, e^{\pm m \int \frac{\alpha\, d\xi}{\sqrt{Rh}}} \left[\sin\left(m \int \frac{\alpha\, d\xi}{\sqrt{Rh}} + c\right) + O(k^{-1})\right],$$

where C and c are arbitrary real constants.

With the introduction of the abbreviations

$$\Psi(\xi) = m \int \frac{\alpha\, d\xi}{\sqrt{Rh}} \tag{33}$$

and

$$F(\xi) = \text{const.} \cdot \sqrt[4]{\frac{R}{r^2h^5}}, \tag{34}$$

these relations take the more compact form

$$\beta_H = CFe^{\pm\Psi} \cos(\Psi + c) \tag{35}$$

and

$$RQ_H = \pm\frac{Eh^2}{2m^2}CFe^{\pm\Psi}\sin(\Psi + c). \tag{36}$$

If loading terms are suppressed in equations (6) to (9), the corresponding terms in the expansions of the remaining quantities of interest are then obtained in the form

$$N_{\xi H} = -Q_H \cot\phi, \tag{37}$$

$$\frac{N_{\theta H}}{Eh} = -\frac{CF}{\sqrt{2}m}\sqrt{\frac{h}{R}}\,e^{\pm\Psi}\sin\left(\Psi + c \pm \frac{\pi}{4}\right), \tag{38}$$

$$\frac{H_H}{Eh} = \mp\frac{CF}{2m^2}\frac{h}{R}\,e^{\pm\Psi}\sin(\Psi + c), \tag{39}$$

$$\frac{M_{\xi H}}{Eh^2} = \mp\frac{CF}{2\sqrt{2}m^3}\sqrt{\frac{h}{R}}\,e^{\pm\Psi}\cos\left(\Psi + c \pm \frac{\pi}{4}\right), \tag{40}$$

$$M_{\theta H} = \nu M_{\xi H}, \tag{41}$$

$$\frac{u_H}{\sqrt{Rh}} = -\frac{CF}{\sqrt{2}m}\frac{r}{R}\,e^{\pm\Psi}\sin\left(\Psi + c \pm \frac{\pi}{4}\right), \tag{42}$$

$$\frac{w_H}{\sqrt{Rh}} = \frac{\alpha}{\sqrt{2}mr'}\left(1 - \frac{r^2}{R^2}\right)CFe^{\pm\Psi}\sin\left(\Psi + c \pm \frac{\pi}{4}\right), \tag{43}$$

where, in recapitulation,

$$m = \sqrt[4]{3(1 - \nu^2)}, \qquad \Psi = m\int\frac{\alpha\,d\xi}{\sqrt{Rh}}, \qquad F = \text{const.}\cdot\sqrt[4]{\frac{R}{r^2h^5}}. \tag{44}$$

7. Particular solutions. Under the present assumption that h, R, and R_1 may vary appreciably only over a distance comparable with a representative dimension of the shell, reference to equation (12a) indicates that, as the bending stiffness tends to zero, with decreasing thickness, the quantity $U = RQ$ also tends to zero (if boundary-layer effects are neglected). The corresponding limiting expression for the rotation β is then given by equation (12b) in the form $\beta = \beta_0$. Thus, for sufficiently thin shells, it may be expected that the expressions $U = 0$, $\beta = \beta_0$ will constitute an approximate particular solution to the inhomogeneous equations (12a, b).

The introduction of these expressions into equations (6) to (9) would then lead to the specification of a state of deformation which approximates a physically permissible one for small h, but which generally does not satisfy the prescribed boundary conditions. By superimposing on this solution an appropriate form of the approximate homogeneous solution, and determining the constants of integration in such a way that the combination satisfies the boundary conditions, an approximate solution to the prescribed problem is then obtained.

The approximate inhomogeneous solution just described is frequently referred to as the "membrane solution." This terminology would seem to imply that the moments obtained by introducing $\beta = \beta_0$ into ($8a$, b) are then to be neglected. Further, some writers have assumed that the membrane rotation β_0 can itself be neglected. Thus it appears to be of some interest to investigate explicitly, in the general case, those circumstances under which the membrane solution provides a degree of approximation consistent, say, with retention of the leading term of the homogeneous developments, and also to determine whether or not the membrane rotation and the corresponding "membrane moments" should then be retained.

In analogy with the homogeneous developments, it may be suggested that expansions for a particular solution of the inhomogeneous equations be assumed in terms of power series in k^{-1}. Thus, if the assumptions

$$(45) \qquad X_P \equiv \sqrt{\frac{rD}{\alpha}}\,\beta_P = \sum_{n=0}^{\infty} A_n(\xi)k^{-n}, \qquad Y_P \equiv \sqrt{\frac{r}{E\alpha h}}\,RQ_P = \sum_{n=0}^{\infty} B_n(\xi)k^{-n}$$

are introduced into equations ($16a$, b), and coefficients of like powers of k^{-1} are formally equated, it follows from a simple calculation that

$$(46) \qquad A_0 = \sqrt{\frac{rD}{\alpha}}\,\beta_0$$

and

$$(47) \qquad \begin{aligned} A_n &= 0, \qquad n \neq 4m \\ B_n &= 0, \qquad n \neq 4m + 2 \end{aligned} \qquad (m = 0, 1, \cdots).$$

Hence the desired expansions are of the type

$$(48) \qquad \beta_P = \beta_0 + \sqrt{\frac{\alpha}{rD}} \sum_{m=1}^{\infty} a_m(\xi)k^{-4m}, \qquad RQ_P = \sqrt{\frac{E\alpha h}{r}} \sum_{m=0}^{\infty} b_m(\xi)k^{-2-4m},$$

and recurrence formulas for the coefficients can be obtained in the form

$$(49) \qquad b_m = \frac{a_m'' - \theta_1 a_m}{2\Phi^2}, \qquad a_{m+1} = -\frac{b_m'' - \theta_2 b_m}{2\Phi^2},$$

where

$$(50) \qquad a_0 = \sqrt{\frac{rD}{\alpha}}\,\beta_0.$$

If now it is assumed that the distributed surface loading may vary appreciably only over a distance comparable with a representative dimension of the shell, it follows that ξ derivatives of β_0 are not large relative to β_0, and hence coefficients of successive powers of k^{-1} in (49) are in general of the same order of magnitude. It follows that in this case one has

$$(51a) \qquad \beta_P = \beta_0[1 + O(k^{-4})],$$

and

$$(51b) \qquad Q_P = \frac{D}{\alpha^2}\left[\sqrt{\frac{\alpha}{rD}}\left(\sqrt{\frac{rD}{\alpha}}\,\beta_0\right)'' - \theta_1\beta_0\right][1 + O(k^{-4})].$$

Thus, with the stated assumptions concerning the rate of variation of h, R, R_1, and of the surface loading, the approximation $\beta_P = \beta_0$ is valid within a fractional error of order k^{-4}. Also, the approximation $Q_P = 0$ then neglects a resultant of the order of magnitude $k^{-4}Eh\beta_0$.

On the other hand, if appreciable fractional changes in surface load intensity could occur over a distance of order $\sqrt{Rh}$, the fractional error in β_P could be of order k^{-2}, while the neglected shear resultant could be of order of magnitude $k^{-2}Eh\beta_0$.

In order to investigate the importance of the omission of terms of such magnitudes, it is necessary to consider their effects on the stresses and deformations in the shell.

8. Order of magnitude of membrane solutions. In the sequel, it is supposed explicitly that appreciable changes in the surface load intensity may occur only over a distance comparable with R. With the convenient identification

$$\alpha = R, \tag{52}$$

it then follows that ξ differentiation does not in general alter the order of magnitude of the loading functions. Shells and portions of shells in which $\csc \phi$ is large of the order $\sqrt{R/h}$ are to be excluded from consideration.

In this case, if the surface loading is of order of magnitude p, the following order-of-magnitude relations follow from equations (10), (14), and (51):

$$V = O(Rp), \tag{53}$$

$$\beta_P = O\left(\frac{R}{h}\frac{p}{E}\right), \tag{54}$$

$$U_P = RQ_P = O(h^2p). \tag{55}$$

Reference to equations (7*a*, *b*) then shows that the contributions of Q_P to the resultants N_ξ and N_θ are in general of order k^{-4} with respect to the contributions of V. It thus follows that, in the cases considered here, the approximation $Q_P = 0$ is acceptable if terms of relative order k^{-4}, *i.e.*, of relative order h^2/R^2, are to be neglected.

With this result, the orders of magnitude of the relevant membrane expressions can be obtained from equations (6) to (9) as follows:

$$\begin{gathered}\beta_P = O\left(\frac{Rp}{hE}\right), \qquad u_P = O\left(\frac{R^2p}{hE}\right), \\ H_P, N_{\xi P}, N_{\theta P} = O(Rp), \\ M_{\xi P}, M_{\theta P} = O(h^2p).\end{gathered} \tag{56}$$

9. Edge stresses and deformations. An inspection of equations (35) to (43) reveals the fact that, if only leading terms in the homogeneous developments are retained, the following relations exist:

$$H_H = \frac{Eh^2}{2m^2 r}\,\beta_H \pm \frac{E}{m}\,\frac{\sqrt{Rh^3}}{r^2}\,u_H, \tag{57}$$

$$M_{\xi H} = \mp\frac{Eh^2}{2m^3}\sqrt{\frac{h}{R}}\,\beta_H - \frac{Eh^2}{2m^2 r}\,u_H, \tag{58}$$

$$N_{\xi H} = \frac{r'}{\alpha}\,H_H, \qquad N_{\theta H} = \frac{Eh}{r}\,u_H, \qquad M_{\theta H} = \nu M_{\xi H}. \tag{59}$$

Equations (57) and (58) can also be resolved to give

$$u_H = \pm\frac{2m}{E}\,\frac{r^2}{\sqrt{Rh^3}}\,H_H + \frac{2m^2 r}{Eh^2}\,M_{\xi H}, \tag{60}$$

$$\beta_H = -\frac{2m^2 r}{Eh^2}\,H_H \mp \frac{4m^3}{Eh^2}\sqrt{\frac{R}{h}}\,M_{\xi H}. \tag{61}$$

In these equations, the upper ambiguous sign refers to those solutions which increase with ξ toward an outer boundary, while the lower sign refers to solutions which increase in the opposite direction. It is considered remarkable that these relations hold everywhere, and not only along an edge.

Attention is restricted in the following to the cases in which either the shell is closed at an apex, in which case only the former solutions are relevant, or the boundaries are sufficiently separated that the latter solutions are of negligible magnitude at the outer boundary. In such cases, only the upper signs in (57) to (61) need be considered when only values near the outer boundary are of interest.

In those cases when surface loading is absent, the subscript H may be suppressed. Equations (57) and (58) then determine edge stress resultants and couples corresponding to prescribed edge deformations, while equations (60) and (61) determine edge deformations corresponding to prescribed edge forces and couples, except for errors of relative order k^{-1}.

In the more general case of surface loading, at a clamped edge one must have $\beta = 0$ and $u = 0$. Here H_H and $M_{\xi H}$ may be replaced, respectively, by $H - H_P$ and $M_\xi - M_{\xi P}$, and β_H and u_H, respectively, by $-\beta_P$ and $-u_P$, if only edge values at the outer boundary are considered, to give the following relations between the edge values:

$$H = H_P - \frac{Eh^2}{2m^2 r}\,\beta_P - \frac{E}{m}\,\frac{\sqrt{Rh^3}}{r^2}\,u_P, \tag{62}$$

$$M_\xi = M_{\xi P} + \frac{Eh^2}{2m^3}\sqrt{\frac{h}{R}}\,\beta_P + \frac{Eh^2}{2m^2 r}\,u_P, \tag{63}$$

$$N_\xi = N_{\xi P} + \frac{r'}{\alpha}\,(H - H_P), \qquad N_\theta = N_{\theta P} - \frac{Eh}{r}\,u_P; \tag{64}$$

$$M_\theta = M_{\theta P} + \nu(M_\xi - M_{\xi P}). \tag{65}$$

Thus the boundary stress resultants and couples are expressed explicitly in terms of boundary values of membrane expressions. By introducing the results of (56) into these relations, it is then possible to estimate the importance of the various membrane expressions in determining the stresses at a clamped edge.

Such an investigation shows that the direct and bending stresses at a clamped edge are in general of the order of magnitude k^2p, where k is proportional to $\sqrt{R/h}$. The contribution of β_P to the meridional direct stress is then of relative order k^{-2}, while its contribution to the maximum bending stress is of relative order k^{-1}. The function β_P is not involved in the determination of the circumferential direct stress. The contribution of the "membrane moment" $M_{\xi P}$ to the bending stress is of relative order k^{-2}.

A similar analysis of the deformations at an unrestrained edge ($H = M_\xi = 0$) shows that the radial displacement of the edge is in general of the order k^3Rp/E, while the edge rotation is of the order k^4p/E. The contribution of β_P to the total rotation is of relative order k^{-2}, while that of the membrane moment is of relative order k^{-3}; the contribution of $M_{\xi P}$ to the radial displacement is also of relative order k^{-3}.

The relative importance of the membrane rotation and moment increases with large edge values of csc ϕ, *i.e.*, for shallow shells.

In the special case of pure radial loading, the membrane resultants H_P and $N_{\xi P}$ vanish. Thus here the preceding discussion must be modified to the extent that the meridional direct stress at a clamped edge is then of the order kp, in general, so that the contribution of β_P is then of relative order k^{-1}. Also, the radial displacement of an unrestrained edge is then generally of the order k^2Rp/E, so that the contribution of $M_{\xi P}$ is then of relative order k^{-2}. The dominant contribution to the rotation of an unrestrained edge is of order k^2p/E, due to β_P, while the effect of $M_{\xi P}$ is then of relative order k^{-1}.

10. Conclusion. From the preceding results, it may be inferred that as long as only the leading term in the homogeneous developments is retained, one should, for consistency, neglect the membrane rotation as well as the corresponding membrane moments, if the loading intensity does not vary appreciably percentagewise over a distance of the order $\sqrt{Rh}$. A more important conclusion consists in the observation that, if the membrane rotation is indeed of importance, then it is not consistent to retain only the leading term in the homogeneous developments, but at least the following term (of relative order k^{-1}) must also be taken into account.

It should be mentioned further that the terms of order k^{-2} and greater in the homogeneous solution are of doubtful validity and would presumably be modified if deformations due to the transverse normal and shear stresses were taken into account in the basic formulation of the problem.

Modifications of the preceding discussion which take into account the possibility of load variation over a distance of order $\sqrt{Rh}$ are readily carried out. Also, as was pointed out by E. Reissner [6], certain cases, in which the geometric characteristics of the shell vary appreciably over such a distance, can also be treated by separating the functions θ_1 and θ_2 appearing in equations (16*a*, *b*) into two parts, one of which will then contain k^2 as a factor. Problems of this type are of particular interest in practice, because of the desirability of rather abruptly increasing the thickness of the shell over that neighborhood of the edge in which the localized bending effects occur.

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SOME RECENT APPLICATIONS OF THE THEORY OF FINITE ELASTIC DEFORMATION

BY

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1. Introduction. In recent years a good deal of work has been done on the theory of finite elastic deformation. It has been applied to various problems which cannot be dealt with by the classical theory of small deformation. It has qualitatively predicted a yield point. It has combined in one condition the two rival hypotheses of elastic failure: principal maximum stress hypothesis and maximum shear stress or maximum principal stress difference hypothesis. It has proved the existence of Bauschinger effect by giving the result that yield stress in compression can be several times that in tension. It has shown how problems pertaining to india rubber shells and tubes turned inside out can be dealt with. When applied to problems in vibrations, it shows that the ordinary vertical oscillations of a particle attached to a light elastic thread are not simple harmonic, and that finite longitudinal and torsional waves cannot be propagated without change of form. It shows that the large difference between the static and dynamic values for elastic bodies like india rubber is in a good measure due to the use of the small-strain theory. Its application gives axial stresses, neglected in the classical theory, in cylinders subjected to large torsional shifts. In compressibility experiments, it gives very good results even for pressures ranging from 2,000 to 20,000 atm. These and other such results of the theory are discussed in this paper.

2. Components of strain. In the theory of finite deformation, the strain components can be described by employing the coordinates of a particle either in the strained state as the independent variables or in the unstrained state as the independent variables. The first is called the Eulerian method and the second is called the Lagrangian. For actual applications of the theory, the importance of using the Eulerian method has been stressed by B. R. Seth [1], F. D. Murnaghan [2], and Ernst-August Deuker [3]. D. Panov [4], P. M. Riz [5], and N. V. Zvolinsky [6] have made a number of applications and have compared their results with that of Seth [1]. The Lagrangian method has been adopted by E. and F. Cosserat [7], M. and L. Brillouin [8], M. A. Biot [9], and recently by R. S. Rivlin [10] and A. Gleysal [11]. A good account of this viewpoint is given by A. Signorini [12]. Angelo Tonolo [13], Umberto Cisotti [14], and R. Kappus [15] have also given results in both these systems. In general, we know the boundary displacements or the boundary tractions in the strained state in which actual observations have to be made. Further, the stress tensor T_{ii} in the body stress equations of equilibrium

$$T_{ij,i} + F_j = 0 \tag{1}$$

is defined in terms of Eulerian coordinates, and if these equations have to be used, the strain tensor must also be referred to the same system. This inconsistency is present in the small-strain theory, though in actual applications it does not affect its results. Thus, as has been done even by Rivlin [10] in discussing the torsion of a circular cylinder, the Eulerian method is to be preferred in applications of the theory of finite strains to actual problems.

In terms of the displacement vector u_i, the finite components of strain are given by

$$2\epsilon_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}. \tag{2}$$

In Lagrangian coordinates, their values are

$$2\eta_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}. \tag{3}$$

As in the small-strain theory, the strain tensor may be decomposed into a strain energy deviation tensor and a strain energy dilatation tensor. This has been attempted by R. Moufang [16] whose results have been criticized by H. Richter [17].

3. Consistency equations. In the strained state, an element of length ds is given by

$$ds^2 = g_{\alpha\beta}dx^\alpha\, dx^\beta, \tag{4}$$

where

$$g_{\alpha\beta} = \delta_{\alpha\beta} - 2\epsilon_{\alpha\beta}. \tag{5}$$

In the unstrained state,

$$(ds_0)^2 = ds_i ds_i. \tag{6}$$

As equation (4) can be reduced to equation (6), the condition for flat-space time is satisfied. The vanishing of the Schwarz–Christoffel tensor is a necessary and sufficient condition for flat-space time. Hence

$$\{\mu\sigma, \alpha\}\, [\alpha\gamma, \epsilon] - \{\mu\gamma, \alpha\}\, [\alpha, \epsilon] - \frac{\partial}{\partial x_\gamma}\, [\mu\sigma, \epsilon] - \frac{\partial}{\partial x_\sigma}\, [\mu\gamma, \epsilon] = 0, \tag{7}$$

where

$$\{\mu\gamma, \sigma\} = \frac{1}{2}\, g^{\sigma\gamma}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\gamma} + \frac{\partial g_{\gamma\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\gamma}}{\partial x_\lambda}\right), \tag{8}$$

$$[\mu\gamma, \sigma] = \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\gamma} + \frac{\partial g_{\gamma\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\gamma}}{\partial x_\sigma}\right). \tag{9}$$

In three dimensions, (7) has only the six components (1212), (2323), (3131), (3112), (1223), and (2331). The vanishing of these constitutes the six consistency equations. Three of them are of the type

$$\frac{\partial^2 g_{11}}{\partial y^2} + \frac{\partial^2 g_{22}}{\partial x^2} - \frac{2\partial^2 g_{12}}{\partial x\, \partial y} = 2\{12, \alpha\}[12, \alpha] - 2\{11, \alpha\}[22, \alpha], \tag{10}$$

and three of the type

$$\frac{\partial^2 g_{11}}{\partial y\, \partial z} + \frac{\partial}{\partial x}\left(\frac{\partial g_{23}}{\partial x} - \frac{\partial g_{31}}{\partial y} - \frac{\partial g_{12}}{\partial z}\right) = 2\{31, \alpha\}[12, \alpha] - 2\{23, \alpha\}[1, \alpha]. \tag{11}$$

These can be simplified if we neglect terms of a degree higher than the second in the strain components [18].

4. Stress-strain relations. A variety of stress-strain relations have been suggested. These will be found in the papers of W. Voigt [19], Seth [1], Murnaghan [2], Rivlin [10], Reiner [20], Swainger [21], Signorini [12], Gleysal [11], Riz [5], Zvolinsky [6], Tolotti [22], Sternberg [23], and others. Seth assumes a linear stress-strain tensor relation which amounts to retaining terms only of the second degree in the invariants I_1 and I_2 in the strain energy function W. In the usual notation, it gives

$$T_{ij} = \lambda \delta_{ij} I_1 + 2\mu \epsilon_{ij}. \tag{12}$$

Like Voigt, who used only the small strain, Murnaghan [2] says that at least terms of the third degree should be kept in W. If l, m, n are the new elastic constants, we get

$$\rho_0 W = \tfrac{1}{2}(\lambda + 2\mu) I_1^2 - 2\mu I_2 + l I_1^3 + m I_1 I_2 + n I_3, \tag{13}$$

ρ_0 being the density in the unstrained state. If ρ is the density in the strained state,

$$\frac{\rho}{\rho_0} = (1 - 2I_1 + 4I_2 - I_3)^{\frac{1}{2}} = 1 - I_1 + \cdots. \tag{14}$$

The stresses are derived from W, by the formulas

$$T_{ij} = \rho\left[\frac{\partial W}{\partial \epsilon_{ij}} - 2\epsilon_{\alpha i} \frac{\partial W}{\partial \epsilon_{\alpha j}}\right], \tag{15}$$

and hence we get

$$\begin{aligned} T_{ij} = {} & [I_1 + (3l + m - \lambda) I_1 + m I_2]\delta_{ij} \\ & + [2\mu - (m + 2\lambda + 2\mu) I_1]\epsilon_{ij} - 4\mu \epsilon_{\alpha i} \epsilon_{\alpha j} + n I_3 \zeta_{ij}, \end{aligned} \tag{16}$$

ζ_{ij} being the tensor reciprocal of ϵ_{ij}, so that $I_3 \zeta_{ij} = \partial I_3/\partial \epsilon_{ij}$. Putting $l = 0$, $m = 0$, $n = 0$, we get relations of the type

$$T_{xx} = \lambda I_1(1 - I_1) + 2[\mu - (\lambda + \mu) I_1]\epsilon_{xx} - 4\mu(\epsilon_{xx}^2 + \epsilon_{xy}^2 + \epsilon_{xz}^2), \tag{17}$$

$$T_{xy} = 2[\mu - (\lambda + \mu) I_1]\epsilon_{xy} - 4\mu(\epsilon_{xx}\epsilon_{xy} + \epsilon_{xy}\epsilon_{yy} + \epsilon_{xz}\epsilon_{yz}). \tag{18}$$

Zvolinsky [6] and Riz [5] take it as an experimental fact that, for finite elastic deformations, the principal stresses are connected with the principal elongations by the linear law of the ordinary theory. If e_{xx} is an elongation (strain) of the small-strain theory, we get from (2)

$$\text{(19)} \qquad \frac{\partial u}{\partial x} = 1 - \sqrt{1 - 2\epsilon_{xx}} = \frac{\partial u}{\partial x'}\left(1 - \frac{\partial u}{\partial x}\right) = e_{xx}\left(1 - \frac{\partial u}{\partial x}\right),$$

(x', y', z') being the coordinates in the unstrained state. Thus,

$$\text{(20)} \qquad \begin{aligned} e_{xx} &= \frac{1 - \sqrt{1 - 2\epsilon_{xx}}}{\sqrt{1 - 2\epsilon_{xx}}} \\ &= \epsilon_{xx} + \tfrac{3}{2}\epsilon_{xx}^2, \end{aligned}$$

if powers higher than the second are neglected. On the assumption of the linear law obeyed by e_{xx} and the corresponding principal stresses, they take the strain energy as

$$\text{(21)} \qquad \rho_0 W = \tfrac{1}{2}\lambda\left(\sum e_{xx}\right)^2 + \mu \sum e_{xx}^2,$$

and making use of the invariants I_1, I_2, I_3, they get

$$\text{(22)} \qquad \rho_0 W = \tfrac{1}{2}(\lambda + 2\mu)I_1^2 - 2\mu I_2 + lI_1^3 + mI_1I_2 + nI_3,$$

where l, m, n are given by

$$\text{(23)} \qquad l = \tfrac{3}{2}(\lambda + 2\mu) = 3\mu\frac{1 - \sigma}{1 - 2\sigma}, \qquad m = -3\mu\frac{1 - 4\sigma}{1 - 2\sigma}, \qquad n = 9\mu,$$

σ being Poisson's ratio. Using these values of l, m, n, and the stress-strain relations (16), they get stresses in the form

$$\text{(24)} \qquad T_{xx} = \lambda I_1 + [2\mu + (\lambda - 2\mu)I_1]\epsilon_{xx} + \tfrac{1}{2}\lambda I_1^2 - 3\lambda I_2 + 5\mu(\epsilon_{xx}^2 + \epsilon_{xy}^2 + \epsilon_{xz}^2),$$

$$\text{(25)} \qquad T_{xy} = [2\mu - (\lambda - 2\mu)I_1]\epsilon_{xy} + 5\mu[\epsilon_{xy}(\epsilon_{xx} + \epsilon_{yy}) + \epsilon_{xz}\epsilon_{yz}].$$

The derivation of $\rho_0 W$ in (21) is open to objection. It implies the wrong assumption that the virtual work done in finite strain is to be obtained in the same manner as in the small-strain theory. Further, the assumption that even in finite strain the tension is proportional to the ordinary stretch is not true, as has been shown both by Seth [1] and Murnaghan [2].

From the expression for T_{ij} given in equations (17), (18), (24), and (25), it is clear that in general the mathematics of the problem is bound to become very complicated in actual applications. Murnaghan [2] treats the case of longitudinal tension and hydrostatic pressure, which give quite simple results. In their applications to the torsion and flexure problems, this difficulty leads Voigt [19], Panov [4], Riz [5], Zvolinsky [6], Biot [9], Rivlin [10], and others to consider the displacements given by the ordinary theory as a first approximation. For example, in dealing with the torsion problem they all follow Voigt in neglecting all powers of τ higher than the second, τ being the twist per unit length. For finite displacements, this method is obviously unsatisfactory. A great deal of the advantage gained in using (13) and (14), instead of (12), is lost in these approximations. In fact they cannot get any change in the torsional rigidity, which is found to be of the order τ^3 [1]. Again, as has been pointed out by Murnaghan [2] himself, the invariants I_1, I_2, I_3 are small. In the case of uniform hydrostatic pressure, I_1 varies between -0.030 and -0.225, I_2 between 0.001 and 0.0169, and I_3 between $-0.1 \cdot 10^{-5}$ and $-0.42 \cdot 10^{-3}$. Further, if the theory is applied to highly elastic incompressible bodies, as has been done by Rivlin [10], the third-order terms in W can be safely neglected. Thus, at the expense of a little accuracy, the linear law in (12) can be adopted. As regards quantitative results, in all laws the constants have to be fitted in consistent with experimental results. All we can expect from a mathematical treatment of the present nature is that it should give good qualitative results, not given by the ordinary theory, and good quantitative results where only comparative ratios are required.

It may be pointed out that some writers, in their treatment of incompressible solids for which $\sigma = \frac{1}{2}$, seem to take the limiting value of λI_1 in (12) for $\lambda \to \infty$ and $I_1 \to 0$, as zero. This need not be the case. All that we can take is that this limit should be finite.

Sternberg [23] uses the form of W given in (13) but assumes the deformation to be small. This amounts to neglecting the second-order space derivatives of the displacement in the strain components and retaining them in the third-degree terms of the strain energy function.

5. Simple tension. If we use the stress-strain relation given in equation (12), we get the components of displacement for a simple tension T as [1]

$$u = x\left[1 - \left(1 + \frac{2\sigma T}{E}\right)^{\frac{1}{2}}\right], \qquad v = y\left[1 - \left(1 + \frac{2\sigma T}{E}\right)^{\frac{1}{2}}\right], \tag{26}$$
$$w = z\left[1 - \left(1 - \frac{2T}{E}\right)^{\frac{1}{2}}\right],$$

σ being the Poisson ratio and E the Young's modulus. If s is the ordinary stretch, we get

$$2T = E\left[1 - \frac{1}{(1 + s)^2}\right]. \tag{27}$$

In Fig. 1, T has been plotted against s. This tension-stretch curve shows that the yield point, where $s \to \infty$, is $T = E/2$. Taking a few more terms, Murnaghan [2] finds that the maximum value of T, which does not correspond to infinite elongation, is $T = E(\lambda + \mu)/4(2\lambda + 3\mu)$. For steel ($\lambda = 1.5\mu$), it gives $T = E/6$. But this is not the yield stress. Both these values are much higher than those found in actual practice. Between A and B the tension-stretch curve is sensibly a straight line, and hence the ordinary small-strain theory holds good in this range. Between B and C, the stretch increases faster than between A and B and at a varying rate. Thus, to get quantitative results, we should attach a meaning to E in formula (27) different from the ordinary Young's modulus. What it should be is to be determined from experimental results. For steel, the yield stress is of the order $30 \cdot 10^3$ lb. per sq. in., and the Young's modulus is of the order $30 \cdot 10^6$ lb. per sq. in. Thus E should be replaced by

$$E_0 = \tfrac{1}{2}10^{-3}E$$

in (27), and then the order of the yield stress will be the same as is found in many elastic materials. This amounts to decreasing E in the ratio $10^{-3}:2$. Similar remarks will apply to the modulus of compression k which, in the range B to C, should be decreased in the ratio $10^{-3}:2$. In like manner, the rigidity modulus should also be decreased. σ, the Poisson ratio, obviously remains unaltered.

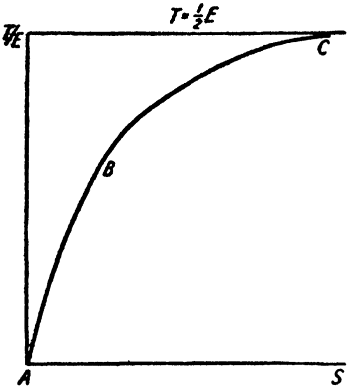

FIG. 1

Having fixed the value of E_0 whose exact value will vary from specimen to specimen, the formula (27) can be expected to give pretty good results. If greater accuracy is required, it can be written in the form

$$T = E_0(s - a^2s^2), \tag{28}$$

and the constant a^2, which is always positive, can be determined from experimental results. This form, which does not use finite values of the displacement, is unsuitable for exhibiting the yield point and leads one to the same result as obtained by Murnaghan.[1]

For india rubber, the application of relation (27) in place of $T = Es$ shows that the ordinary static value of its Young's modulus is much less than what it should be. The dynamical value of its Young's modulus is almost twice the static value, and hence the use of (27) brings the two values much closer together.

If l_0, l are the unstretched and stretched lengths of an india rubber string hanging under its own weight, which is w per unit length of the unstretched string, the ordinary linear law and the one given in (27) give the extension produced in it as $\frac{1}{2}wl_0^2/E_1$ and $\frac{1}{2}wl^2/E_2$, respectively. These values show that, if $T = Es$ is to be used, E should be increased in the ratio l^2/l_0^2, which is always greater than 1, and can be as high as 2 for $l = l_0\sqrt{2}$. See [24].

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[1] It should be pointed out that the tension-stretch formula obtained by L. R. G. Treloar [26] for elastic materials like rubber is

$$T = G\left[1 + s - \frac{1}{(1 + s)^2}\right],$$

where T is the tension measured per unit area of the unstrained section and G a constant. This is similar to (27). His tension-elongation curve is very much like the curve given in Fig. 1.

It is clear from both (27) and (28) that for compression, when s is negative, the value of T_c is not the same as for tension T. In fact for the same absolute value of s, T_c is greater than T. Moreover, E_0 will also now have a different value. If desired, a formula similar to (28) can be constructed for the case of compression. It will be of the form

$$T_c = E_1(s + a^2s^2), \tag{29}$$

T_c and s denoting the absolute values.

6. Hydrostatic pressure. The method suggested above should be used in all results given by the finite strain theory in order that good quantitative agreement may be obtained with actual observation. In fact, Murnaghan uses this method for comparing his results with those of Bridgeman [25] obtained experimentally for the compressibility of sodium in the range 2,000 to 20,000 atm., and finds very good agreement between the two results. If we use (12), we get

$$p = \frac{3}{2}k\left[\frac{1}{(1-s)^2} - 1\right], \tag{30}$$

p being the pressure, s the radial contraction, and k the modulus of compression. As already suggested, the order of $3k$ should be

$$3k \cdot 10^{-3} \cdot 2 = 15 \cdot 10^4, \tag{31}$$

k being of the order $25 \cdot 10^6$ in the range A to B. Putting (30) in the form

$$p = 3k(s + a^2s^2 + b^2s^3), \tag{32}$$

we see that it reduces to the formula used by Murnaghan if $3k$ is of the order $15 \cdot 10^4$. As is to be expected from the remarks we have made, Murnaghan finds that the value of $3k$ decreases very rapidly as we pass from one range of p to a higher one. For ordinary pressures in the range B to C (Fig. 1), formula (30) with the new value of k can be expected to give good results.

7. Pure flexure. If an initially plane rectangular plate is bent into a circular cylinder of inner radius a and outer radius b by couples applied at the edges, the components of displacement [1] are given by

$$u = x - f(r), \qquad v = y - A\theta, \qquad w = 0, \tag{33}$$

where

$$\frac{df}{dr} = \left(\frac{l}{r^c} - \frac{A^2}{r^2}\right)^{\frac{1}{2}}, \qquad c = \frac{1 - 2\sigma}{1 - \sigma}, \tag{34}$$

$$A^2 = \frac{(2-c)(ab)^{2-c}(a^c - b^c)}{c(a^{2-c} - b^{2-c})}, \qquad l = \frac{(2-c)(a^2 - b^2)}{a^{2-c} - b^{2-c}}. \tag{35}$$

The stress components are

$$T_{rr} = \lambda + \mu + \frac{A^2}{r^2} - \frac{(\lambda + 2\mu)l}{2r^c}, \qquad T_{r\theta} = 0, \tag{36}$$

$$T_{\theta\theta} = \lambda + \mu - \frac{\mu A^2}{r^2} - \frac{1}{2}\frac{1}{r^c}, \tag{37}$$

and the bending couples M_1 and M_2 applied to the straight and curved edges, respectively, are

$$M_1 = \mu \left[\tfrac{1}{2}(b^2 - a^2) - \frac{(2 - c)(\log b/a)(ab)^{2-c}(a^c - b^c)}{c(a^{2-c} - b^{2-c})} \right], \tag{38}$$

$$M_2 = -\lambda \left[\tfrac{1}{3}(b^3 - a^3) + \tfrac{1}{2} \frac{2 - c}{3 - c} (a^2 - b^2) \frac{a^{3-c} - b^{3-c}}{a^{2-c} - b^{2-c}} \right]. \tag{39}$$

The unstretched longitudinal plane is $r = A$ and is not the middle plane, as has been assumed by Swainger [27] in solving this problem. In finite pure flexure, the assumption made by Swainger is not true, and hence his solution, which is even more complicated than ours, needs modification.

H. J. Barten [28], K. E. Bisshopp and D. C. Drucker [29], and H. D. Conway [30] have discussed large deflection of cantilever beams by taking the usual expression D/ρ for the bending moments, where $D = EI$ or $EI/(1 - \sigma^2)$ according as it is a long beam or a wide beam. They retain the second-order terms in ρ but do not consider a modification of the bending moment formula. Their treatment is the same as that for the elastica. The second-order terms in the bending moments can be obtained [31] from (38) and (39) in the form

$$M_1 = \frac{D}{\rho} \left[1 - \tfrac{1}{15}(2 - 6c + c^2) \frac{h^2}{\rho^2} \right], \tag{40}$$

$$M_2 = -\frac{D}{\rho} \left[1 - \tfrac{1}{60}(13 - 7c)(1 - c) \frac{h^2}{\rho^2} \right], \tag{41}$$

$2h$ being the width of the beam. If ρ has a finite value, terms of the type h^2/ρ^3 should not be neglected unless h is a very small quantity.

Let P be the concentrated load at the free end and $\rho = ds/d\theta$. If $\theta = \theta_0$ at the free end where $\rho = 0$, the equation to determine ρ at any point is found to be given by

$$(2 - 6c + c^2) \frac{h^2}{\rho^4} - \frac{10}{\rho^2} + 20 \frac{P}{D} (\sin \theta - \sin \theta_0) = 0. \tag{42}$$

This reduces to the equation

$$\frac{1}{\rho^2} = \frac{2P}{D} (\sin \theta - \sin \theta_0) \tag{43}$$

given by Barten and others if h^2/ρ^4 is neglected.

8. Torsion and flexure. For large torsional shifts Saint-Venant gives the components of displacement [1] as

$$\begin{aligned} u &= -y \sin \alpha z + x (1 - \cos \alpha z), \\ v &= x \sin \alpha z + y (1 - \cos \alpha z), \\ w &= \alpha \phi(x, y). \end{aligned} \tag{44}$$

These assume that there is no radial displacement in a section of the cylinder. For a finite twist in a circular cylinder, a radial displacement should be assumed, and then the components of displacement become

$$\begin{aligned} u &= y\beta \sin \alpha z + x\,(1 - \beta \cos \alpha z), \\ v &= x\beta \sin \alpha z + y\,(1 - \beta \cos \alpha z), \\ w &= dz, \end{aligned} \tag{45}$$

β being a function of $r(r = x^2 + y^2)$ only.

Using (12) and assuming that the resultant of the axial stresses also vanishes over the plane ends, we get

$$\beta^2 = 1 - \tfrac{1}{4}(1 - c)\alpha^2(r^2 - a^2) + \frac{1}{6\cdot 8^2}(1 - c)^2\alpha^4[r^4(12 - c) - 24a^2r^2 + (8 + c)r^4] + \cdots, \tag{46}$$

$$d = \tfrac{1}{4}\alpha^2a^2 + \tfrac{1}{96}(5 - 2c)\alpha^4a^4 + \cdots, \tag{47}$$

where $c = (1 - 2\sigma)/(1 - \sigma)$, and a is the radius of the cylinder in the strained state. The torsional couple M is given by

$$M = \tfrac{1}{2}\mu\pi\alpha a^4\left[1 + \tfrac{1}{12}(1 - c)\alpha^2a^2 - \frac{(1 - c)^2}{12\cdot 16}(1 - \tfrac{1}{4}c)\alpha^4a^4 + \cdots\right]. \tag{48}$$

Riz, Panov, Rivlin, and others do not use the finite components of strain as given in (45). Riz uses (45) for a circular cylinder but assumes β to be of the form $1 + \alpha^2 f(r)$, and thus cannot get any change in the torsional couple M, which is of the third degree in α. Panov uses terms only of the second degree in α for discussing the torsion of an elliptic cylinder. Rivlin takes Saint-Venant values and assumes no radial displacement. His calculations are also confined only to the second degree in α, and so he also finds that M is unaltered. For $\alpha a = \frac{1}{2}\pi$, this correction for M is as much as 6.41 per cent.

The stress components corresponding to the displacement (45) are found [1] to be

$$T_{rr} = \lambda\Delta + \mu(1 - \beta^2) - \mu\left(\beta'^2 + \frac{2\beta\beta'}{r}\right)r^2, \qquad T_{r\theta} = 0, \tag{49}$$

$$T_{\theta\theta} = \lambda\Delta + \mu(1 - \beta^2), \tag{50}$$

$$T_{zz} = \lambda\Delta + \mu(2d - d^2 - \beta^2\alpha^2r^2), \tag{51}$$

$$T_{\theta z} = \mu\alpha r\beta^2, \qquad T_{rz} = 0, \tag{52}$$

where

$$\Delta = 1 + \beta^2 + d - \tfrac{1}{2}d^2 - \frac{1}{2}\left(\beta^2\alpha^2 + \beta'^2 + \frac{2\beta\beta'}{r}\right)r^2, \qquad \beta' = \frac{d\beta}{dr}.$$

If we make $\lambda \to \infty$ and $\sigma \to \frac{1}{2}$ so that $E = 3\mu$, $c\lambda = \frac{2}{3}E$, and retain terms up to the second degree in α, we get

$$\lambda\Delta = \tfrac{1}{6}\alpha^2(a^2 - 2r^2), \tag{53}$$

$$\begin{aligned} T_{rr} &= \tfrac{1}{12}E\alpha^2(a^2 - r^2), \\ T_{\theta\theta} &= \tfrac{1}{12}E\alpha^2(a^2 - r^2) - \tfrac{1}{3}E\alpha^2 r^2, \\ T_{zz} &= \tfrac{1}{3}E\alpha^2(a^2 - 2r^2), \\ T_{\theta z} &= \tfrac{1}{3}E\alpha r. \end{aligned} \tag{54}$$

There is a serious drawback in the solutions of Kappus, Rivlin, and Riz. The first two take $\beta = 1$ and the third $\beta = 1 + \alpha^2 f(r)$, and thus both of them have no arbitrary constant at their disposal to adjust the normal traction on the plane ends. They should have taken $\beta = A_0[1 + \alpha^2 f(r)]$, if powers of α higher than the second are neglected. Then it is found that the statement that the torsion of the cylinder is accompanied by a varying normal traction of the type $(ar^2 - b)$, where a and b are known constants, acting on the plane ends, has no meanings. They can adjust the resultant traction to any desired value. In our solution we have assumed it to be zero. In his torsion solution for the elliptic cylinder, Panov introduces a constant to satisfy the condition of no traction over the plane ends. From (54) we see that in our solution the unstretched longitudinal fiber is given by $r = a/\sqrt{2}$, terms higher than α^2 being neglected.

If a_0 is the radius of the cylinder before strain, we get

$$a_0 = a\left[1 - \frac{1}{3\cdot 8^2}(1 - c)^2\alpha^4 a^4 + \cdots\right], \tag{55}$$

so that the change in the radius of the cylinder is of the order α^4. No wonder the approximate results mentioned above do not given any change in a_0. Riz gets a change in a_0 of the order α^2 without satisfying the boundary condition on the plane ends. He fails to notice that we have taken the plane ends to be free from traction. If a finite value is taken for the traction on the plane ends, the change in a_0 will be of the order α^2. Again, for a first approximation, this finite value of the traction and the corresponding extension d are found from (47) and (54) to be proportional to α^2. Rivlin verifies this by experiment with a rubber cylinder.

The flexure problem has been discussed by Riz, A. J. Gorgidze, and A. K. Ruchadze [32], [33]. Gorgidze and Ruchadze obtain the first-order torsion-bending interaction term.

9. Shells and tubes under uniform pressure. For the symmetrical deformation of a thick spherical shell subjected to uniform internal and external pressures, we find [34] that the radial displacement and the stress components are given by

$$u_r = (1 - P)r, \tag{56}$$

$$T_{rr} = \tfrac{1}{2}(3\lambda + 2\mu)(1 - P^2) - \tfrac{1}{2}(\lambda + 2\mu)P^2V(V + 2), \tag{57}$$

$$T_{\theta\theta} = T_{\phi\phi} = \tfrac{1}{2}(3\lambda + 2\mu)(1 - P^2) - \tfrac{1}{2}\lambda P^2V(V + 2), \tag{58}$$

where P is a function of $r(r^2 = x^2 + y^2 + z^2)$ only, $V = r\,dP/P\,dr$, and the equation giving V is

$$(59)\quad \log r = \tfrac{1}{6}\log(V^2 + 2FV + 3) - \tfrac{1}{2}\log V^2 - \frac{3 - F}{3\sqrt{3 - F^2}}\tan^{-1}\left[\frac{V + F}{\sqrt{3 - F^2}}\right] + k,$$

where k is a constant of integration and $F = (\lambda + 3\mu)/(\lambda + 2\mu)$. Using these equations, Shepherd and Seth [34] have discussed a number of cases which cannot be dealt with by the small-strain theory. Their results show that the stresses T_{rr} and $T_{\theta\theta}$ should satisfy the inequality

$$(60)\qquad -\frac{1 - 2\sigma}{1 - \sigma} < \frac{2(T_{rr} - T_{\theta\theta})}{3\lambda + 2\mu - 2T_{rr}} < \frac{1 - 2\sigma}{2\sigma}.$$

Similar results have been obtained for a circular cylindrical tube. They get

$$(61)\qquad u_r = (1 - Q)r,$$

$$(62)\qquad w = \alpha z,$$

$$(63)\qquad T_{rr} = \lambda\Delta + \mu[1 - Q^2(U + 1)^2],$$

$$(64)\qquad T_{\theta\theta} = \lambda\Delta + \mu(1 - Q^2),$$

$$(65)\qquad T_{zz} = \lambda\Delta + \mu[1 - (\alpha - 1)^2],$$

where

$$\Delta = \tfrac{1}{2}[3 - (\alpha - 1)^2 - Q^2\{1 + (U + 1)^2\}], \qquad \frac{r}{Q}\frac{dQ}{dr} = U,$$

and

$$(66)\quad \log r = \frac{1}{4}\log(U^2 + 2GU + 2) - \frac{1}{4}\log U^2 - \frac{2 - G}{2\sqrt{2 - G^2}}\tan^{-1}\left[\frac{U + G}{\sqrt{2 - G^2}}\right] + k,$$

where $G = (2\lambda + 5\mu)/2(\lambda + 2\mu)$. The inequality corresponding to (60) is

$$(67)\qquad -\frac{1 - 2\sigma}{1 - \sigma} < \frac{T_{rr} - T_{\theta\theta}}{\lambda + \mu - T_{rr}} < \frac{1 - 2\sigma}{\sigma}.$$

If r' is the value of r in the unstrained state, we get

$$(68)\qquad \frac{dr'}{dr} = Q(1 + U),$$

which is negative if $U < -1$. Putting $\sigma = 0.49$, they have discussed the case of an india rubber shell turned inside out with its plane ends and curved boundary free from traction. They find that the percentage increase in the external radius of the tube is about 3 per cent, and this is approximately verified by experiment.

The finite deformation of a hollow sphere subjected to internal pressure has also been discussed by F. A. Bahšiyan [35].

10. Elastic failure. The limitations imposed on the stress components by the inequalities given in (60) and (67) give rise to an important criterion of elastic failure [36]. According to the maximum stress hypothesis

$$p_1 - p_3 = k, \qquad \text{const.,} \tag{69}$$

where the principal stresses p_1, p_2, p_3 are in descending order of magnitude. Putting $p_3 = 0$, and then $p_1 = -p_2$, we get the ratio between the yield point in shear and the yield point in tension as

$$\xi = 0.5. \tag{70}$$

Von Mises–Hencky hypothesis for elastic failure can be written as

$$(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 = k, \tag{71}$$

and proceeding as above, we get

$$\xi = 0.577. \tag{72}$$

A feature common to all such theories of elastic failure is that the criterion of failure is unaltered by a reversal of the sign of stress. But it is well known that materials show yield stresses in compression several times those in tension. Again, the values of ξ in (70) and (72) do not depend upon the elastic constants. In actual practice, they always depend upon σ. The inequalities in (60) and (67) give a criterion which does away with these drawbacks and contains in itself the two rival hypotheses of elastic failure.

Taking the upper limit, we get for extension

$$p_1 - p_3 + \frac{1}{2}\frac{1 - 2\sigma}{1 + \sigma}(p_1 + p_2 + p_3) = \frac{3}{4}\frac{E}{1 + \sigma}, \tag{73}$$

If p and S are the yield stresses in tension and shear, we get[2]

$$p = \frac{1}{2}E, \qquad S = \frac{3}{8}\frac{E}{1 + \sigma}, \tag{74}$$

and hence

$$\xi = \frac{3}{4}\frac{1}{1 + \sigma}. \tag{75}$$

In these formulas, the modified form of E suggested in Sec. 5 is understood. Most of the materials with which common experiments are done have σ varying between 0.25 and 0.3. In such cases, we get

$$0.577 < \xi < 0.6, \tag{76}$$

which is in very good agreement with tensile tests.

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[2] These relations show that S, the yield stress in shear, varies between $\frac{1}{2}\mu_0$ and $\frac{3}{4}\mu_0$, when μ_0, the reduced value of μ, is of order

$$12 \cdot 10^6 \cdot 2 \cdot 10^{-3} = 24 \cdot 10^3.$$

Thus we have $12 \cdot 10^3 \leqq S \leqq 18 \cdot 10^3$. This agrees with the experimental values of S for many materials.

For contraction, we easily get

$$\xi = \frac{3}{2} \cdot \frac{\sigma}{1 + \sigma}, \tag{77}$$

and, for $0.25 < \sigma < 0.3$, we have $0.3 < \xi < 0.346$.

From (67), assuming that $T_{rr} - T_{zz}$ is the maximum principal stress difference, we get for tension[3]

$$p = \frac{E}{2(1 + \sigma)}, \qquad S = \frac{E}{4(1 + \sigma)}, \qquad \xi = 0.5, \tag{78}$$

and for contraction

$$p = \frac{E(1 - \sigma)}{6\sigma^2}, \qquad S = \frac{E(1 - \sigma)}{4\sigma(1 + \sigma)}, \qquad \xi = \frac{3\sigma}{2(1 + \sigma)}. \tag{79}$$

Thus, if k is the ratio of yield stress in compression to yield stress in tension,

$$k = \frac{1 - \sigma^2}{3\sigma^2}. \tag{80}$$

For concrete, $\sigma = 0.2$, and we get $r = 8$, which is in good agreement with experimental results. Thus we see that in all cases the criterion of failure can be written as

$$p_1 - p_3 + c(p_1 + p_2 + p_3) = k, \tag{81}$$

where c and k are suitable constants. This is similar to that suggested by J. J. Guest [37], with this difference that he uses only $p_1 + p_3$ and not the average volumetric stress. Guest says that the form (81) is found to be suitable even for (1) combined stress and (2) anisotropic materials.

11. Vibrational problems. If we use the tension stretch formula (12) for the vertical vibration of a mass m attached to a light elastic string, we get [38] the relation between the time t and the displacement as

$$t = 2\left(\frac{ml}{E}\right)^{\frac{1}{2}} \left(\frac{l'}{l}\right)^{\frac{3}{2}} (\sec \theta)^{\frac{1}{2}} E(k, \phi). \tag{82}$$

In (82) l is the unstretched length, l' the stretched length in the position of equilibrium, $k = \sin \theta = (\alpha - \beta)/\alpha$ is the modulus of the elliptic function $E(k, \phi)$ of the second order, and α, β are the values of the vertical distance y measured from the fixed point where the velocity of m vanishes. The value of y is

$$y = \alpha(1 - k^2 \sin^2 \phi), \tag{83}$$

and the complete period is

$$T = 4\left(\frac{ml}{E}\right)^{\frac{1}{2}} \left(\frac{l'}{l}\right)^{\frac{3}{2}} (\sec \theta)^{\frac{1}{2}} E_0, \tag{84}$$

E_0 being the complete integral of the second kind. In the ordinary simple harmonic motion,

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[3] See footnote 2, page 78.

$$T_0 = 2\pi \left(\frac{ml}{E}\right)^{\frac{1}{2}}. \tag{85}$$

Thus the actual motion is not a simple harmonic. Table I gives the values of the amplitude $l(\alpha - \beta) = l' \sin\theta \tan\theta$ and the ratio $Tl^{\frac{1}{2}}/T_0 l'^{\frac{1}{2}}$ for different values of θ.

TABLE I

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°
$(2/\pi)E_0\ (\sec\theta)^{\frac{1}{2}}$	1	1.000	1.001	1.004	1.014	1.037	1.090	1.217	1.589
$\sin\theta\tan\theta$	0	0.031	0.124	0.289	0.539	0.913	1.500	2.582	5.585

It is obvious that the period does not vary much with the amplitude as long as it is not greater than 0.539 l'. This is what Baker [39] found in his experiments with rubber. Treloar [26] also confirms this result.

For finite longitudinal vibrations of strings and rods, the relation (12) gives [40] the differential equation

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \xi/\partial x^2}{(1 + \partial\xi/\partial x)^3}, \tag{86}$$

which is exactly the general equation for long waves. We get a similar equation for the vibrations of a rod struck at one end or suddenly loaded.

For torsional vibrations, on using the value of the torsional couple given in (48), we get the differential equation

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{\mu}{\rho}\frac{\partial^2 \theta}{\partial z^2}\left[1 + \frac{1}{4}(1 - c)a^2 \left(\frac{\partial \theta}{\partial z}\right)^2\right], \tag{87}$$

which is similar to (86) if powers of $\partial\xi/\partial x$ beyond the second are neglected. In such cases, we get [41] the velocity u given by

$$u = f[x' - t(a_1 u + a_2)], \tag{88}$$

where $x' = x + \xi$, and a_1 and a_2 are suitable constants. Thus, as is to be expected, finite waves cannot be propagated without change of form.

12. Nonisotropic bodies. In two papers Seth [42] has discussed the following problems of finite strain pertaining to nonisotropic elastic bodies: (1) simple tension, (2) hydrostatic pressure, (3) pure cylindrical flexure of a rectangular plate, (4) torsion of a right circular cylinder, and (5) symmetrical strain in shells and tubes under uniform pressure.

In the case of simple tension, shear is also produced unless the cylinder possesses three orthogonal planes of symmetry. Two sets of values for the components of displacement have been obtained and, it is possible, one of them may correspond to aeolotropic materials like pyrites which expand slightly in a lateral direction when extended in the direction of a principal axis.

In pure cylindrical flexure, the plate is assumed to be transversely isotropic.

For the moment of the couple M_1 applied to the straight edges, we get Table II. a and b are the inner and outer radii of the bent cylindrical plate. $t = b/a$ and the constant c_0 is given in terms of the elastic constants c_{11} and c_{66} by the relation $c_0 = 2\, c_{66}/c_{11}$.

TABLE II

t	$c_0 = 0$		$c_0 = 0.5$		$c_0 = 1$		$c_0 = 1.5$		$c_0 = 2$	
	$\frac{10^2 M_1}{c_{66}a^2}$	$\frac{10^2 M_1}{c_{11}a^2}$	$\frac{10^2 M_1}{c_{66}a^2}$	$\frac{10^2 M_1}{c_{11}a^2}$	$\frac{10^2 M_1}{c_{66}a^2}$	$\frac{10^2 M_1}{c_{66}a^2}$	$\frac{10^2 M_1}{c_{66}a^2}$	$\frac{10^2 M_1}{c_{11}a^2}$	$\frac{10^2 M_1}{c_{66}a^2}$	$\frac{10^2 M_1}{c_{11}a^2}$
1	0	0	0	0	0	0	0	0	0	0
1.1	0.5	0	0.1	0.025	0.01	0.005	0.008	0.005	0	0
1.2	1.5	0	0.3	0.075	0.12	0.06	0.06	0.045	0	0
1.3	3.1	0	0.6	0.150	0.40	0.20	0.20	0.15	0	0
1.4	5.2	0	1.4	0.350	0.89	0.45	0.45	0.33	0	0
1.5	7.7	0	2.5	0.630	1.68	0.84	0.85	0.63	0	0

If M_2 is the couple applied to the curved edges, it is found that $M_2 \to 0$ as $c_0 \to 0$. Let us assume that c_{66} remains finite and $c_{11} \to \infty$ such that the stresses remain finite, as we do in the case of an incompressible isotropic body. From Table II we see that M_1 remains finite. Thus it should be possible to bend such a plate into a cylindrical shape by applying couples only to the straight edges.

The stretching, bending, and torsion of bars, composed of different materials having the same Poisson ratio but different elastic moduli, have been discussed by Gorgidze [43] and Ruchadze [44]. L. M. Milne-Thompson [45] has also discussed finite deformation in aeolotropic bodies.

13. Miscellaneous problems. Seth [46] has discussed the finite strain in a rotating shaft. He discusses the two cases when (1) the length of the cylinder is maintained constant and (2) the resultant longitudinal tension vanishes over the plane ends. He takes the particular case when the angular velocity ω is connected with the elastic constants by the relation

$$\rho a^2 \omega^2 = \frac{\mu(2 - c)(10 + c)}{c(5 - c)}, \tag{89}$$

ρ being the density, a the strained radius, and $c = (1-2\sigma)/(1-\sigma)$. Comparing his results with those given by the small-strain theory, he finds that the difference between them can be as much as 10 per cent.

S. T. Newing and W. M. Shepherd [47] have discussed finite strain dislocation solutions and have obtained a number of interesting results. In particular, they obtain numerical results for a plate of constant initial curvature, bent by edge couples so that its curvature is either increased or decreased. They find that the couple required to produce a given decrease in curvature is less than that calculated from the small-strain theory.

The case of a spherical shell turned inside out has been discussed by Seth [48]. He finds that the increase in its external radius is about 2 per cent.

The following problems concerning incompressible cylinders and spheres have also been treated by Seth [49]: (1) symmetrical bending of cylinders acted upon by a body force, with special reference to a rotating shaft and (2) a sphere acted upon by a body force, with special reference to a gravitating sphere.

He has compared his results with those given by the small-strain theory.

14. Summary. It is shown that theory of finite elastic deformation can be profitably used in the range where the linear elasticity ceases and the yield point begins. In this range it is known that the strain increases faster than it does in the linear domain, and at a varying rate. Hence it is suggested that in this range the elastic constants E, μ, k should be different from those used in the small-strain theory. A tentative formulation decreasing them in the ratio $10^{-3}:2$ gives good quantitative results.

The recent work on finite strain by a number of workers has been critically examined, and the drawbacks in some of their solutions have been pointed out.

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ELASTIC STABILITY OF THE FACINGS OF SANDWICH COLUMNS[1]

BY

H. W. MARCH

1. Introduction. The problem of the wrinkling of the facings of a sandwich column under compressive end load is essentially the problem of the stability of two elastically supported sheets, of which the elastic support of one is influenced to a greater or less extent by the presence of the other. In the mathematical treatment of the problem, it has been customary to consider types of instability of the facings involving sinusoidal wave patterns whose wavelength is so chosen that the compressive wrinkling stress is a minimum for the type of instability considered. The two types of instability that are commonly considered are the following:

(a) *Antisymmetrical wrinkling.* The sinusoidal wrinkles of the two facings are in phase as shown in Fig. 1*a*.

(b) *Symmetrical wrinkling.* The sinusoidal wrinkles differ in phase by 180°, so that the facings are displaced symmetrically with respect to the undistorted middle plane of the core as shown in Fig. 1*b*.

A theoretical study of the wrinkling of the facings of sandwich columns of isotropic materials in compression was made by Gough, Elam, and deBruyne [5], who in order to obtain relatively simple formulas for the critical compressive stress, introduced certain simplifying assumptions.[2] It was noted by Williams, Leggett, and Hopkins [12] that these assumptions led to incorrect results for wrinkles of long wavelengths. They introduced more suitable boundary conditions and developed formulas for sandwich columns with either isotropic or orthotropic cores. For columns with isotropic cores and antisymmetrical wrinkling, they found that, except for wrinkles of long wavelengths where buckling of the column as a whole rather than wrinkling may be expected, the formulas of Gough, Elam, and deBruyne led to results in satisfactory agreement with those obtained by their own formulas. The formulas of Williams, Leggett, and Hopkins are exceedingly complicated and are not suited to the numerical computations required for an extensive series of sandwich constructions. Cox [3] made an investigation of sandwich columns with orthotropic cores, in which he introduced parameters that are very well suited to the problem at hand, and gave an extended discussion of both symmetrical and antisymmetrical wrinkling. His formulas for antisymmetrical wrinkling are in agreement with those of Williams, Leggett, and Hopkins. For sandwich columns with thick cores, he derived satisfactory approximate formulas. Van der Neut [9] undertook to improve the boundary conditions used by Gough, Elam, and deBruyne but, as pointed out by Bijlaard [1], he tacitly retained one of their assumptions. Bijlaard made the necessary modifications to remove this restriction and carried out an analysis for columns with isotropic facings and core, the result of which is in essential agreement with that of Williams, Leggett, and Hopkins. Hoff and Mautner [6] considered symmetrical and other types of wrinkling, and used assumptions similar to those of Gough, Elam, and deBruyne. An estimate of the destabilizing effect of the compressive stress in the core was made by Williams, Leggett, and Hopkins [12].

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[1] This paper presents a survey of the work done on this problem by a group at the Forest Products Laboratory in cooperation with the U.S. Air Force–Navy–Civil Subcommittee on Aircraft Design Criteria under the supervision of the Aircraft Committee of the Munitions Board. The group consisted of the following men: K. Boller, engineer; W. S. Ericksen, mathematician; C. B. Norris, engineer; C. B. Smith, mathematician (now associate professor of mathematics at the University of Florida); and the author, a Forest Products Laboratory mathematician and also professor of mathematics at the University of Wisconsin. The U.S. Forest Products Laboratory is maintained at Madison, Wis., in cooperation with the University of Wisconsin.

[2] These authors refer in particular to two earlier papers, one by Biot [2] and the other by Reissner [8], which were concerned with the problem of a beam on an infinite elastic foundation. In the latter paper one of the problems considered is that of the buckling of an infinitely long strut resting on a series of equally spaced rigid supports and restrained on one side by a two-dimensionally infinite medium of the same width as the strut.

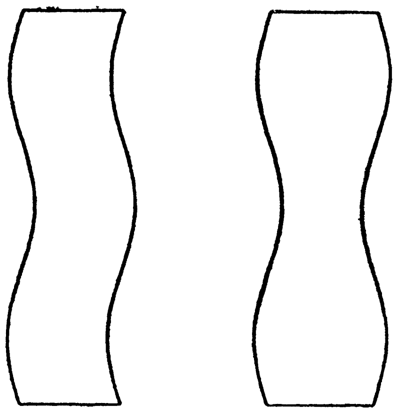
(*a*) (*b*)

FIG. 1. Types of wrinkling instability: (*a*) antisymmetrical; (*b*) symmetrical.

The following authors should also be mentioned:

Wan [10], who derived formulas applicable to sandwich constructions with cores whose elastic properties are assumed to satisfy certain restrictions; Williams [11], whose aim was to present simplified methods for the use of the designer; Goodier [4] and Neuber [7], who derived highly complicated formulas to include the destabilizing effect of the compressive stress in the core. Both Wan and Williams considered the effect of initial irregularities.

In the present paper, in which the stability of the facings of sandwich columns with orthotropic cores and facings is studied, the assumptions made by Gough, Elam, and deBruyne [5] in treating sandwich columns of isotropic materials are used. It is found that the results for a wide range of materials can be presented in a limited number of curves by making use of certain dimensionless parameters. The importance of this will be perceived when the great variation in the ratios of the many elastic constants of orthotropic materials is considered. Indeed, it is found that for a wide range of sandwich constructions of practical interest the results can be expressed with sufficient accuracy by a single curve. For an extended group of constructions, the formulas of the present paper are shown to lead to results that are in satisfactory agreement with those of the more exact but complicated formulas obtained by Williams, Leggett, and Hopkins [12] and by Cox [3].

A tentative method is mentioned for taking into account the reduction in observed wrinkling stress associated with initial irregularities in the facings. A few curves are presented that show a comparison of the results of tests and of theory.

2. Orientation of axes. The coordinate axes (Fig. 2) are so oriented that the xz plane is the middle surface of the core in its undeformed state. The axis of x is parallel to the direction of loading. The thickness of the core is denoted by c and that of each facing by f.

3. Assumed form of wrinkles. It is assumed that, when the facings become unstable under compressive end load, the planes $y = \pm c/2$ are deformed into sinusoidal surfaces and that in these planes the displacement in the direction of loading may be neglected.

The components of displacement in the core parallel to the axes of x and y will be denoted u and v, respectively. The following equations express the assumptions made with respect to these components at the junctions of the core and facings.

$$
\begin{aligned}
u_1 &= u_2 = 0,\\
v_1 &= a \sin \frac{\pi x}{L},\\
v_2 &= \delta a \sin \frac{\pi x}{L}.
\end{aligned}
\tag{1}
$$

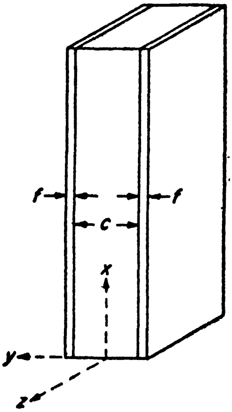

Fig. 2. Section of sandwich column showing orientation of axes.

The subscripts 1 and 2 refer to the surfaces $y = c/2$ and $y = -c/2$, respectively. The factor δ will take the value $+1$ or -1, which will correspond, respectively, to antisymmetrical or to symmetrical wrinkling. The half wavelength L is to be determined so that the compressive load necessary to maintain the deformation (1) is a minimum. The loads on the two facings are assumed to be equal. The assumption that $u = 0$ on the planes $y = \pm c/2$ is a plausible approximation to the actual situation for sinusoidal patterns of short wavelengths. It is justified by the consequent simplification in the analysis and by the agreement of the results obtained with those of a more exact analysis. For columns buckling as a whole, the assumption cannot be made, but for such columns the buckling loads can be found rather simply by taking into account the effect of shear deformation in the core.

4. Stress-strain relations of the core. The core of the sandwich column is assumed to be orthotropic, with the orthotropic axes coinciding with the axes of reference. The problem is reduced to one in two dimensions by assuming either a state of plane stress or of plane strain, as a given situation may require. In either state, the relation between the components of stress and strain can be written in the forms[3]

$$\begin{aligned} e_{xx} &= a_{11}X_x - a_{12}Y_y \\ e_{yy} &= -a_{12}X_x + a_{22}Y_y \\ e_{xy} &= \frac{X_y}{\mu_{xy}} \end{aligned} \tag{2}$$

For the state of plane stress,

$$a_{11} = \frac{1}{E_x}, \qquad a_{12} = \frac{\sigma_{xy}}{E_x}, \qquad a_{22} = \frac{1}{E_y} \tag{3}$$

and for plane strain,

$$\begin{aligned} a_{11} = \frac{1 - \sigma_{xz}\sigma_{zx}}{E_x}, \qquad a_{12} = \frac{\sigma_{xy} + \sigma_{xz}\sigma_{zy}}{E_x} \\ a_{22} = \frac{1 - \sigma_{zy}\sigma_{yz}}{E_y}. \end{aligned} \tag{4}$$

In these relations E_x and E_y are Young's moduli, μ_{xy} is a modulus of rigidity, and $\sigma_{xz}, \sigma_{zx}, \cdots$ are Poisson's ratios.

5. The stress function and the stress components. A stress function $F(x, y)$ is introduced such that

$$X_x = \frac{\partial^2 F}{\partial y^2}, \qquad Y_y = \frac{\partial^2 F}{\partial x^2}, \qquad X_y = -\frac{\partial^2 F}{\partial x\, \partial y}. \tag{5}$$

It is found by using equations (2) and the condition of compatibility of the strain components e_{xx}, e_{yy}, and e_{xy} that $F(x, y)$ satisfies the differential equation

$$a_{22}\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{\mu_{xy}} - 2a_{12}\right)\frac{\partial^4 F}{\partial x^2\, \partial y^2} + a_{11}\frac{\partial^4 F}{\partial y^4} = 0. \tag{6}$$

By the transformation

$$\eta = \epsilon y, \tag{7}$$

where

$$\epsilon = \left(\frac{a_{22}}{a_{11}}\right)^{\frac{1}{4}}, \tag{8}$$

this equation takes the more convenient form

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[3] Love's notation for the components of stress and strain is used. See A. E. H. Love, *The mathematical theory of elasticity*, Cambridge University Press, London. For the general relations between the components of stress and strain in orthotropic materials, see, for example, Forest Products Laboratory Report 1312, p. 34.

$$\frac{\partial^4 F}{\partial x^4} + 2\kappa \frac{\partial^4 F}{\partial x^2\, \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0, \tag{9}$$

where

$$\kappa = \frac{1}{2\sqrt{a_{11}\, a_{22}}}\left(\frac{1}{\mu_{xy}} - 2a_{12}\right). \tag{10}$$

A suitable solution of (9) is

$$F = (A_1 \sinh \nu y + A_2 \sinh \bar{\nu} y + B_1 \cosh \nu y + B_2 \cosh \bar{\nu} y) \sin \frac{\pi x}{L}, \tag{11}$$

where

$$\nu = \frac{\pi \epsilon \alpha}{L}, \qquad \bar{\nu} = \frac{\pi \epsilon \bar{\alpha}}{L}, \tag{12}$$

$$\alpha = \alpha_1 + \alpha_2, \qquad \bar{\alpha} = \alpha_1 - \alpha_2, \tag{13}$$

$$\alpha_1 = \sqrt{\frac{\kappa + 1}{2}}, \qquad \alpha_2 = \sqrt{\frac{\kappa - 1}{2}}. \tag{14}$$

From (5) and (11), expressions for the stress components are obtained, and these are substituted in (2) to obtain the expressions for the components of strain. The integration of these expressions yields the components of displacement u and v in the core. The requirement that these components satisfy the conditions (1) on the surfaces $y = \pm c/2$ determines the following values for the coefficients in equation (11):

$$\begin{aligned}
A_1 &= \frac{(1 + \delta) a\, \epsilon\, L \bar{g}_1 \sinh (\bar{\nu} c/2)}{2\pi[g_1 \bar{g}_2 \alpha \sinh (\nu c/2) \cosh (\bar{\nu} c/2) - \bar{g}_1 g_2 \bar{\alpha} \sinh (\bar{\nu} c/2) \cosh (\nu c/2)]},\\
A_2 &= \bar{A}_1,\\
B_1 &= \frac{(1 - \delta) a\, \epsilon\, L \bar{g}_1 \cosh (\bar{\nu} c/2)}{2\pi[g_1 \bar{g}_2 \alpha \sinh (\bar{\nu} c/2) \cosh (\nu c/2) - \bar{g}_1 g_2 \bar{\alpha} \sinh (\nu c/2) \cosh (\bar{\nu} c/2)]},\\
B_2 &= \bar{B}_1,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
g_1 &= a_{11}\epsilon^2\alpha^2 + a_{12}, & g_2 &= a_{12}\epsilon^2\alpha^2 + a_{22},\\
\bar{g}_1 &= a_{11}\epsilon^2\bar{\alpha}^2 + a_{12}, & \bar{g}_2 &= a_{12}\epsilon^2\bar{\alpha}^2 + a_{22},
\end{aligned} \tag{16}$$

and where $\bar{A}_1$ and $\bar{B}_1$ denote the quantities A_1 and B_1, respectively, with each function of α changed to the corresponding function of $\bar{\alpha}$, and vice versa.

From (5), (11), and (15), the following expressions for the stress component Y_y at the surfaces $y = \pm c/2$ are obtained:

$$\begin{aligned}
(Y_y)_{y=c/2} &= a\gamma_1 \sin \frac{\pi x}{L},\\
(Y_y)_{y=-c/2} &= a\gamma_2 \sin \frac{\pi x}{L},
\end{aligned} \tag{17}$$

where

$$(18) \qquad \gamma_i = \frac{\pi\epsilon}{2L}(g_1 - \bar{g}_1)\left[\frac{(-1)^{i+1}(1+\delta)}{g_1\bar{g}_2\alpha \coth(\bar{\nu}c/2) - \bar{g}_1 g_2\bar{\alpha}\coth(\nu c/2)} + \frac{(1-\delta)}{g_1\bar{g}_2\alpha \tanh(\bar{\nu}c/2) - \bar{g}_1 g_2\bar{\alpha}\tanh(\nu c/2)}\right]_{i=1,2}.$$

6. The condition for instability. The differential equation of equilibrium of the facing, $y > c/2$, under a compressive load P per inch of edge of this facing when it has a lateral displacement v_1, is readily found to be

$$(19) \qquad \frac{E_f f^3}{12\lambda_f}\frac{d^4 v_1}{dx^4} + P\frac{d^2 v_1}{dx^2} + (Y_y)_{y=c/2} = 0,$$

where E_f is the Young modulus of the facings measured in a direction parallel to the direction of loading and $\lambda_f = 1 - \sigma_{xy}\sigma_{yx}$ calculated for the facings.

In deriving this equation by considering an element of the facing, the moment about the mid-plane of the facing of the force arising from the shear stress at the face of the core is neglected. This amounts to assuming for this purpose that the core is attached to the mid-plane of the facing.

For the opposite facing, the corresponding equation is

$$(20) \qquad \frac{E_f f^3}{12\lambda_f}\frac{d^4 v_2}{dx^4} + P\frac{d^2 v_2}{dx^2} - (Y_y)_{y=-c/2} = 0.$$

If, however, it is observed from (17) that

$$(Y_y)_{-c/2} = -\delta(Y_y)_{y=c/2},$$

when $\delta = 1$ and also when $\delta = -1$ and that

$$v_2 = \delta v_1,$$

it will be seen that the second of these equations is equivalent to the first when $\delta = +1$ or when $\delta = -1$. Accordingly, only the first will be used.

On substituting in this equation the expressions (1) and (17) for v_1 and $(Y_y)_{y=c/2}$, respectively, an algebraic equation is obtained that can be solved for P, the load per inch of edge of the facing that is necessary to maintain it in the wrinkled state defined by equation (1). It is assumed throughout that an equal load P is acting on the other facing so that the compressive stress $p = P/f$ is the same in both facings.

Before writing this solution for P, extensive changes in notation will be made, including the introduction of certain dimensionless parameters. The combinations (25) and (26) below are due to Cox [3]. Gough, Elam, and deBruyne [5], Williams [11], and Hoff and Mautner [6] used combinations quite similar to the first of those in (24). The second of (24) combined with the first has proved very useful in plotting curves presenting the results of the analysis.

(21) $$p = \frac{P}{f}, \qquad r = \frac{c}{f},$$

(22) $$\zeta = \frac{\epsilon\pi c}{L}, \qquad \nu c = \alpha\zeta, \qquad \bar{\nu} c = \bar{\alpha}\zeta;$$

(23) $$\bar{E} = \left(\frac{E_f \mu_{xy}}{\lambda_f a_{22}}\right)^{\frac{1}{3}}, \qquad \chi = \frac{\mu_{xy}}{\bar{E}},$$

(24) $$p = Q\bar{E}, \qquad r = \frac{q}{\chi},$$

(25) $$E = \frac{1}{(a_{11}a_{22})^{\frac{1}{2}}}, \qquad \sigma = a_{12}E,$$

(26) $$\beta = \frac{\mu_{xy}}{E}.$$

By using these relations, it follows from (10) that

(27) $$\kappa = \frac{E}{2\mu_{xy}} - \sigma = \frac{1}{2\beta} - \sigma;$$

from (8) that

(28) $$a_{11} = \frac{1}{\epsilon^2 E}, \qquad a_{22} = \frac{\epsilon^2}{E};$$

from (16) that

(29) $$g_1 = \frac{\alpha^2 + \sigma}{E},$$

(30) $$g_2 = \frac{\epsilon^2(\alpha^2\sigma + 1)}{E};$$

from (29) and (13) that

(31) $$g_1 - \bar{g}_1 = \frac{4\alpha_1\alpha_2}{E}.$$

From (29) and (30), together with the relation

(32) $$\alpha\bar{\alpha} = 1,$$

it is found after some reduction that

(33) $$g_1\bar{g}_2\alpha = \alpha_1\alpha_2\epsilon^2\left(\frac{G}{\alpha_1} + \frac{H}{\alpha_2}\right),$$

(34) $$\bar{g}_1 g_2\bar{\alpha} = \alpha_1\alpha_2\epsilon^2\left(-\frac{G}{\alpha_1} + \frac{H}{\alpha_2}\right),$$

where

$$G = 1 + (1 - \sigma^2)\beta, \tag{35}$$

$$H = 1 - (1 - \sigma^2)\beta. \tag{36}$$

As a result of using the foregoing transformations and making extensive reductions,[4] the expression found from equation (19) for the stress, $p = P/f$, in the facings corresponding to the wrinkled state defined by equation (1) reduces to the equation

$$Q = \frac{\zeta^2\beta}{12q^2} + \frac{2qR}{\zeta}, \tag{37}$$

where

$$R = \frac{(1 + \delta)(\cosh \alpha_1\zeta - \cosh \alpha_2\zeta)}{2[(G \sinh \alpha_1\zeta)/\alpha_1 + (H \sinh \alpha_2\zeta)/\alpha_2]} + \frac{(1 - \delta)(\cosh \alpha_1\zeta + \cosh \alpha_2\zeta)}{2[(G \sinh \alpha_1\zeta)/\alpha_1 - (H \sinh \alpha_2\zeta)/\alpha_2]}. \tag{38}$$

In equation (37), in accordance with equations (21), (22), and (24), Q is proportional to p, the stress in the facings, q to the ratio c/f of the thickness of the core to that of the facings, and ζ to the ratio c/L of the thickness of the core to the half wavelength of the wrinkles in the facings. It is to be noted that the elastic properties of the facings do not appear in the function R in the second term of equation (37).

If the core is isotropic, $\alpha_1 = 1$ and $\alpha_2 = 0$. Formula (37) applies in this case after the limit has been taken as α_2 approaches zero.

7. Critical wrinkling stress. In accordance with equation (24), the stress in the facings p and the ratio r of the thickness of the core to that of the facings are constant multiples of Q and q, respectively, for given core and facings. Hence a relative minimum of p with respect to ζ with r held constant may be found by using the variables Q and ζ with q held constant. The process of determining a minimum of Q with respect to ζ is equivalent to determining the half wavelength L [see (22)] of the sinusoidal pattern requiring the least compressive stress to maintain it.

The condition

$$\frac{\partial Q}{\partial \zeta} = 0 \tag{39}$$

is expressed by the relation

$$q^3 = \frac{\zeta^3\beta}{12\left(R - \zeta\dfrac{dR}{d\zeta}\right)}. \tag{40}$$

By means of this equation, together with (37), Q is defined as a function of q with ζ as a parameter. It can be shown[4] that, corresponding to a value of ζ

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[4] For the details, the reader is referred to Forest Products Laboratory Report 1810.

for which Q has a minimum, the slope of q plotted as a function of ζ [equation (40)] is positive. This test can be applied when there is question as to whether or not the condition (39) determines a minimum of Q as a function of ζ. By using equations (37) and (40), together with the procedure described in the next paragraph, Q can be plotted as a function of q for a series of values of the parameters β and σ. The values of these parameters are determined solely by the elastic properties of the core.

8. Antisymmetrical wrinkling. For antisymmetrical wrinkles, δ in equations (37) and (40) is equal to 1. The values of Q_f, minima of Q with respect to ζ for various values of q, are found as follows: After assigning values to the parameters β and σ, a series of values of ζ are substituted in (40) to find the corresponding values of q. Then pairs of corresponding values of ζ and q are substituted in (37) to find the associated value of Q_f.

Plots of Q_f as a function of q are given in Figs. 3 to 5. The ranges chosen for the parameters β and σ include their values for cores of end-grain balsa, cellular cellulose acetate, cork, and cellular hard rubber.

In the figures, the point at which a given curve ends at the left is called the "terminal" point. The value of q at this point, *viz.*, $q = q_0$, divides the range of q into two parts. On the basis of the present assumptions, a sandwich column that is originally flat and for which $q < q_0$ cannot develop antisymmetrical wrinkles. This is because Q plotted as a function of ζ by equation (37) for such a value of q has no minimum point. When sufficient load is applied, the column will fail in some other way, such as by column buckling, by shear failure in the core, or possibly, by symmetrical wrinkling. On the interval $q > q_0$, antisymmetrical wrinkling is possible and will occur if the corresponding load is less than that associated with some other mode of failure. The existence of a terminal point was recognized by Gough, Elam, and deBruyne. Calculations based on the analyses of Williams, Leggett, and Hopkins and of Cox, who used more exact assumptions, also show the existence of a terminal point.

Analytically, q_0 is the smallest value of q for which Q has a true minimum with respect to ζ. From the present analysis, it is found that for the ranges of β and σ considered herein, this value of q is either identical with

$$\lim_{\zeta \to 0} q = \sqrt[3]{\frac{2}{1 - \sigma^2}}, \tag{41}$$

as obtained from equation (40) or is so close to it that the difference can be neglected. The corresponding value of Q, obtained from (37) with $\delta = 1$, is

$$\lim_{\zeta \to 0} Q = \frac{q}{2} = \frac{1}{2}\sqrt[3]{\frac{2}{1 - \sigma^2}}. \tag{42}$$

In Figs. 3 and 4 the terminal point is located by the coordinates given by (41) and (42). This is also true for the curves $\beta = 0.6$ and $\beta = 0.8$ in Fig. 5. For $\beta = 0.2$ and $\beta = 0.4$ in Fig. 5, the smallest q for which Q has a minimum does not coincide exactly with that given by (41). The greatest discrepancy occurs when $\beta = 0.2$. In such cases, the graphs of Q as a function of ζ have a maximum point

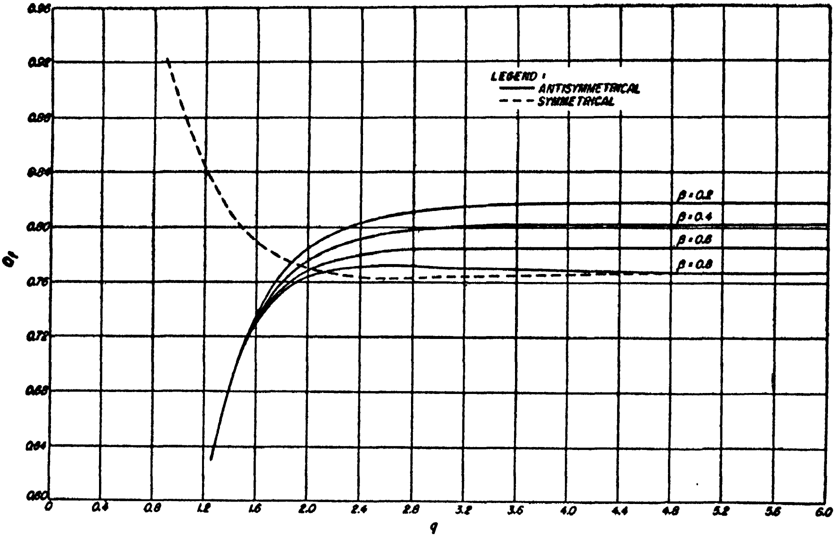

FIG. 3. Q_f vs. q for symmetrical and antisymmetrical wrinkling with $\sigma = 0$, and $\beta = 0.2$, 0.4, 0.6, and 0.8.

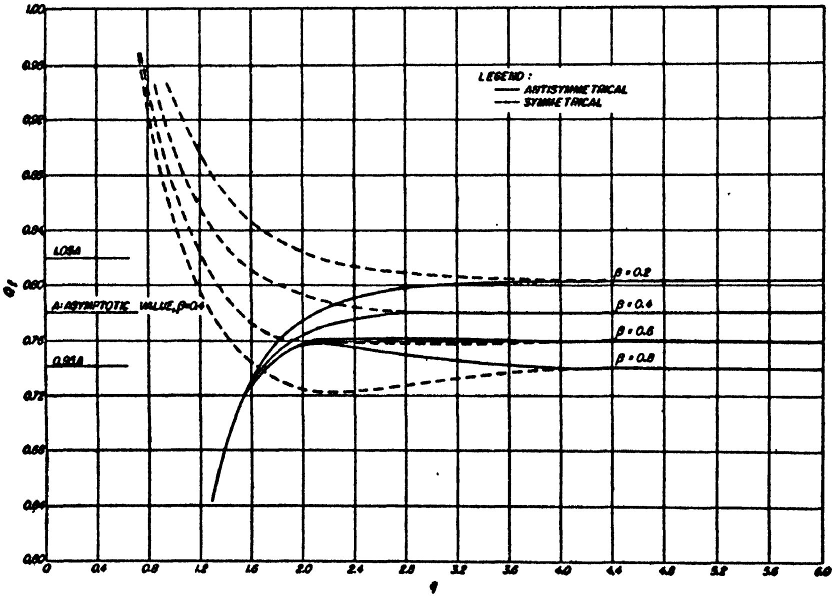

FIG. 4. Q_f vs. q for symmetrical and antisymmetrical wrinkling with $\sigma = 0.25$, and $\beta = 0.2$, 0.4, 0.6, and 0.8.

between $\zeta = 0$ and the minimum point for values q in a narrow range lying above the true terminal point.

Figure 6 and a number of other computations based on the more exact theory of Williams, Leggett, and Hopkins and of Cox indicate that the point determined by (41) and (42) is a good approximation to the true one. The value of ζ at the terminal point, when using the more accurate formulas, is some value of ζ other than zero, as is shown in Fig. 7, but the value of Q_f is approximately that given by (42) and not the value zero that would actually be associated with $\zeta = 0$, that is, $L = \infty$. One is consequently led to conclude that the present theory based on the assumptions of Gough, Elam, and deBruyne errs in the predicted wavelength at failure when q is near the terminal value rather than in the value of Q that is proportional to the stress in the facings.

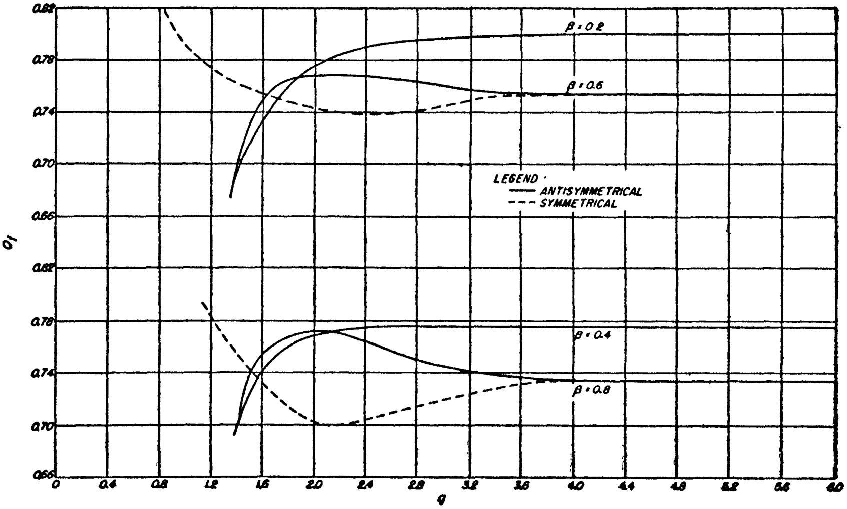

FIG. 5. Q_f vs. q for symmetrical and antisymmetrical wrinkling with $\sigma = 0.5$, and $\beta = 0.2$, 0.4, 0.6, and 0.8.

When ζ is large, it is found from (40) that

$$\frac{q}{\zeta} = \left(\frac{G\beta}{12\alpha_1}\right)^{\frac{1}{3}}. \tag{43}$$

By using this expression, it follows from (37) that

$$\lim_{\zeta \to \infty} Q = \frac{3}{2}\left(\frac{1 + 2(1 - \sigma)\beta}{6G^2}\right)^{\frac{1}{3}}. \tag{44}$$

Reference to Figs. 3 to 5 shows that this asymptotic value of Q is approximately correct even for only moderately large values of q.

Figures 3 to 5 indicate that the variation of Q_f with respect to σ for any given value of q is quite insignificant. Moreover, the variation of Q_f with respect to β is not large. Hence, from a practical point of view, one can choose mean values of both parameters and base prediction of wrinkling stress on a single curve. For example, if $\beta = (\mu_{xy})/E \leqq 0.6$, as it is for the core materials mentioned above, the curve for $\beta = 0.4$ and $\sigma = 0.25$ gives values of Q_f that differ by 5 per cent at most from those given by any curve drawn.

9. Symmetrical wrinkling. For symmetrical wrinkling, curves for Q as a function of q are obtained by eliminating ζ between equations (37) and (40) after setting $\delta = -1$. For this mode of wrinkling, there is no terminal point since Q has a minimum with respect to ζ for any value of q. When q is small, *i.e.*, when the core is thin, the values of Q_f are large and Q_f becomes infinite as q approaches zero. Obviously some other type of failure, such as failure by column buckling or by shear failure in the core, will occur if q is small. As q becomes large, Q_f approaches the asymptotic value given by equation (44), which is the asymptotic value for antisymmetrical wrinkling. For intermediate values of q, the values of Q_f for symmetrical wrinkling may be smaller than those for antisymmetrical wrinkling. This situation exists when β and σ are such that the constant, κ, equations (10) and (27), is less than unity. When β is moderately small, at most equal to 0.6, and σ is less than 0.5, the difference between the two values of Q_f will be small. The difference may be important for certain honeycomb constructions.

If κ is greater than or equal to unity, Q_f for antisymmetrical wrinkling is less than or at most equal to that for symmetrical wrinkling for any value of q for which the former mode of wrinkling is possible.

The plots of Q_f as a function of q for symmetrical wrinkling are shown in Figs. 3 to 5, where the corresponding curves for antisymmetrical wrinkling are also given. In Fig. 4 a curve for symmetrical wrinkling is given for each of the four values of β. In Figs. 3 and 5, however, the curves for symmetrical wrinkling are those for which Q_f, symmetrical, is noticeably less than the corresponding Q_f, antisymmetrical, over a range of values of q. It is seen from the curves that, for the combinations of β and σ considered, Q_f, antisymmetrical, never exceeds Q_f, symmetrical, by a very large amount.

The reader who may be interested in a further discussion of the stresses associated with antisymmetrical and symmetrical wrinkling should consult Cox's paper [3].

10. Discussion of results of a more accurate analysis. The analysis that has been used in this paper is considered to be approximate for three main reasons, viz.:

(*a*) The component u of the displacement at the glue line has been taken to be zero.

(*b*) The core has been considered to be attached directly to the middle surface of the facings in setting up the differential equation of equilibrium of a facing.

(*c*) The destabilizing effect of the compressive stress in the core has been neglected.

These are restrictions that limit the range of applicability of the theory. It is the purpose of this section to discuss these limitations.

In the analyses of Williams, Leggett, and Hopkins [12] and of Cox [3], assumptions (*a*) and (*b*) are not made. The formulas obtained by those authors are very complicated and are thus not suited to calculating an extended series of sandwich constructions. For considering the effect of these two assumptions, the work of Cox, which is in a form particularly suited to the purpose, is chosen as a basis for comparison. In this comparison E_f/λ_f, corresponding to a state of plane strain, will be used in place of E_f wherever Young's modulus of the facings in the direction of loading occurs in Cox's formula.

The formula developed by Cox for antisymmetrical wrinkling (equation 27, reference [3]), after transformation to the variables used in the present report, can be put in the form

$$p_c = Q_\chi \bar{E}, \tag{45}$$

p_c = stress in each facing associated with a given value of

$$Q_\chi = \frac{\zeta^2 \beta}{12 q^2} + \frac{\dfrac{2qR}{\zeta} + \dfrac{\chi^2}{\beta^2}\dfrac{q^2}{\zeta^2} S + \chi T + \dfrac{\chi^2 \zeta}{2q} U}{1 + \dfrac{2q}{\zeta}\dfrac{\chi^2}{\beta} U}, \tag{46}$$

where χ and R are defined by (23) and (38), respectively, and where, using the abbreviation,

$$M = \frac{G \sinh \alpha_1 \zeta}{\alpha_1} + \frac{H \sinh \alpha_2 \zeta}{\alpha_2},$$

$$S = \frac{(\sinh \alpha_1 \zeta)/\alpha_1 - (\sinh \alpha_2 \zeta)/\alpha_2}{M},$$

$$T = \frac{(1 - \sigma)(\sinh \alpha_1 \zeta)/\alpha_1 + (1 + \sigma)(\sinh \alpha_2 \zeta)/\alpha_2}{M},$$

$$U = \frac{\cosh \alpha_1 \zeta + \cosh \alpha_2 \zeta}{M}.$$

Equation (46) reduces to equation (37) when χ, as it appears explicitly in (46), is set equal to zero.

Comparison of the critical wrinkling stress predicted by (46) and by (37) necessitates finding the minimum values of Q_χ and Q for a given q and plotting these minima $Q_{\chi f}$ and Q_f as functions of q. Since the complexity of the expressions involved precludes an analytic investigation, recourse must be had to computations.

Figure 6 shows graphs of Q_f as calculated from (37) and (40) and of $Q_{\chi f}$ for $\chi = 0.05$ and 0.1. In drawing these curves, the values of the parameters β and σ were taken to be 0.4 and 0.25, respectively. For the values of the parameters used in constructing the curves of Fig. 6, it is clear that Q_f (present theory) is less than Q_{xf} (Cox theory). Hence for these values of the parameters, the predictions of the present theory are conservative in comparison with those of a more exact analysis.

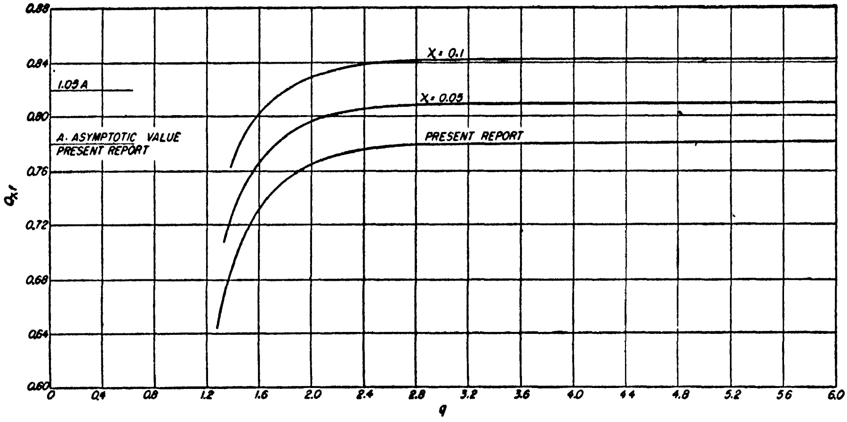

FIG. 6. Comparison of critical values Q_{xf} from the Cox formula with those from the present report for antisymmetrical wrinkling. $\sigma = 0.25, \beta = 0.4$.

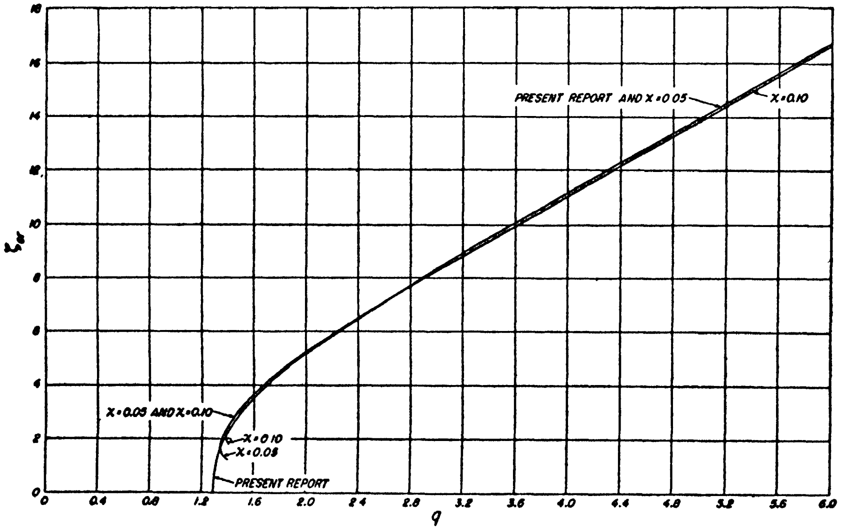

FIG. 7. Comparison of values of ζ_{cr} as obtained by the Cox formula with those obtained by the formula of the present report for antisymmetrical wrinkling. $\sigma = 0.25$; $\beta = 0.4$. NOTE: When ζ is shown as a two-valued function of q, the smaller value corresponds to a relative maximum.

The extreme values of χ used in constructing Fig. 6 include its range for a wide variety of sandwich constructions. This may be seen from Tables I and II.

TABLE I. MECHANICAL PROPERTIES OF CORE MATERIALS USED TO PREDICT WRINKLING STRESSES IN THE FACINGS

CORE MATERIAL	SPECIFIC GRAVITY	COMPRESSION		SHEAR	TENSION	POISSON'S RATIO			
		Modulus of elasticity		Modulus of rigidity, 1,000 p.s.i. μ_{xy}	Maximum stress, p.s.i. t				
		Flatwise, 1,000 p.s.i. E_y	Edgewise, 1,000 p.s.i. E_x			σ_{xy}	σ_{xz}	σ_{zy}	σ_{yx}
Cork[1]	0.35	0.52	1.18	0.33	42	0.136[2]	0.10		0.06
	0.45	1.40	3.55	0.98	51	0.127[2]	0.11		0.05
	0.66	4.89	13.70	3.40	87	0.224[2]	0.21		0.08
Cellular cellulose acetate[1]									
Flatwise	0.10	17.28	4.58	3.99	180	0.09	0.21	0.39	
Edgewise	0.10	23.59	5.44	4.47	180	0.21	0.09		
Balsa wood (end grain)[3]	0.12	536.00	8.01	15.00	880	0.02	0.50	0.02	0.40
Cellular cellulose acetate (flatwise)[3]	0.10	40.70	12.98	3.50	201	0.09	0.21	0.39	
Cellular hard rubber[3]	0.10	27.50	6.95	3.90	240	0.26	0.40		0.19

[1] Average values obtained in this investigation.

[2] Values computed by $\sigma_{xy} = \dfrac{E_x \sigma_{yx}}{E_y}$.

[3] Average values obtained from tables in Forest Products Laboratory Report 1561.

TABLE II. TYPICAL VALUES OF PARAMETERS USED TO PREDICT THE WRINKLING STRESS IN FACINGS

CORE MATERIAL	CORE SPECIFIC GRAVITY	CORE PARAMETERS			SANDWICH PARAMETERS					
					Facing material					
					Aluminum		Steel		Papreg or fiberglas	
		E, 1,000 p.s.i.	β	σ	E, 1,000 p.s i.	χ	E, 1,000 p.s.i.	χ	E, 1,000 p.s.i.	χ
Cork[1]	0.35	0.78	0.421	0.090	12.0[2]	0.0275[2]	17.6	0.0187	7.32	0.0450
	0.45	2.23	0.439	0.080	24.8[2]	0.0395[2]	35.3	0.0278	14.6	0.0671
	0.66	8.19	0.415	0.134	55.0	0.0618	81.0[2]	0.0420[2]	33.7	0.1010
Cellular cellulose acetate[1]										
Flatwise	0.10	8.90	0.448	0.175	88.4	0.0450	131.0[2]	0.0310[2]	54.0	0.0733
Edgewise	0.10	11.34	0.394	0.438	102.0	0.0440	150.0[2]	0.0298[2]	62.3	0.0720
Balsa (end grain)[3]	0.12	65.60	0.229	0.164	431.0[2]	0.0348[2]	636.0	0.0236	264.0[2]	0.0570[2]
Cellular cellulose acetate (flatwise)[3]	0.10	23.00	0.152	0.160	112.0[2]	0.0312[2]	166.0	0.0210	69.8	0.0500
Cellular hard rubber[3]	0.10	13.83	0.282	0.520	102.0[2]	0.0382[2]	151.0	0.0260	62.6	0.0620

[1] Parameters computed from values determined in present investigation.

[2] Sandwich parameters for combinations of facing and core materials which have been tested.

[3] Parameters computed from values taken from Forest Products Laboratory Report 1561.

For large values of q, it can be shown that, approximately,

$$Q_{\chi f} = \frac{3}{2}\left[\frac{1 + 2(1 - \sigma)\beta}{6G^2}\right]^{\frac{1}{3}} + \frac{\chi(1 - \sigma)}{G} + \frac{\chi^2\alpha_1}{2G}\left(\frac{12\alpha_1}{G}\right)^{\frac{1}{3}}. \tag{47}$$

The difference between the asymptotic values of Q_f and $Q_{\chi f}$ can be calculated by this formula for various values of the parameters χ, β, and σ.

In Fig. 8 are shown graphs of the stress in a facing plotted against $\zeta = (\epsilon\pi c)/L$ as calculated by the approximate formula (37) with $\delta = 1$ and by the more exact formula of Cox or of Williams, Leggett, and Hopkins for two values of the ratio, $r = c/f$. The curves drawn for $r = 100$ show the excellent agreement of the results except for long wavelengths, *i.e.*, small ζ. The minimum values of p are nearly equal. The curves for $r = 39$ are drawn to illustrate the meaning of the terminal value of q. The curve obtained from the approximate formula has a horizontal tangent at $\zeta = 0$ and no true minimum point. The exact curve also has no minimum point but has a horizontal inflectional tangent. For smaller values of r, neither curve has a true minimum. For such constructions, antisymmetrical wrinkling does not occur. Since there is no preferred wavelength, failure will be by column buckling or by some other type of failure.

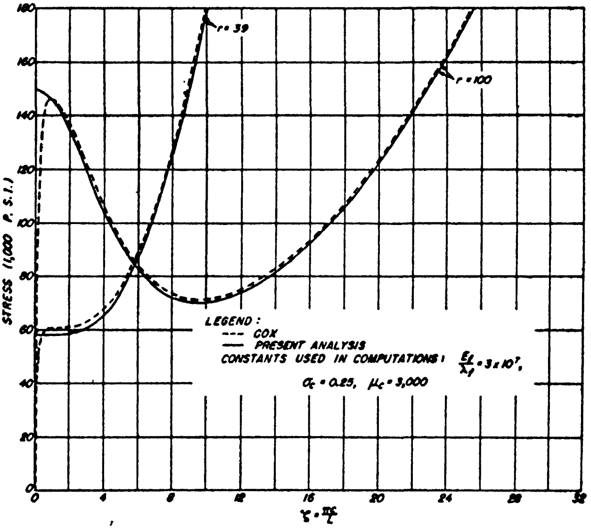

FIG. 8. Comparison of graphs of stress in facings vs. ζ obtained by the Cox formula with those obtained by the formula of the present report. Isotropic facings and core.

Goodier [4] and Neuber [7] investigated the effect of the compressive stress in the core on the stability of the facings. The results are expressed in very complicated formulas. The effect in any given case must be evaluated by extended computations. Neuber made some comparisons with the results of Gough, Elam, and deBruyne for isotropic materials, but they are shown in curves that are not available. On the basis of an estimate made by Williams, Leggett, and Hopkins, it appears that the effect may be expected to be small.

11. The stresses in the core when the faces are not initially flat. In the mathematical treatment of the problem of the wrinkling of a flat sandwich, the core is considered to be free of all stresses except the direct stresses associated with the compressive load until the point of instability is reached. When the facings are not originally flat, the irregularities initially present change in form and magnitude with the increase in load, and the core is stressed locally throughout the loading process. The problem of predicting failure under these circumstances is one of determining the load at which the core or bond is stressed to its ultimate at some load lower than that associated with instability of the facings.

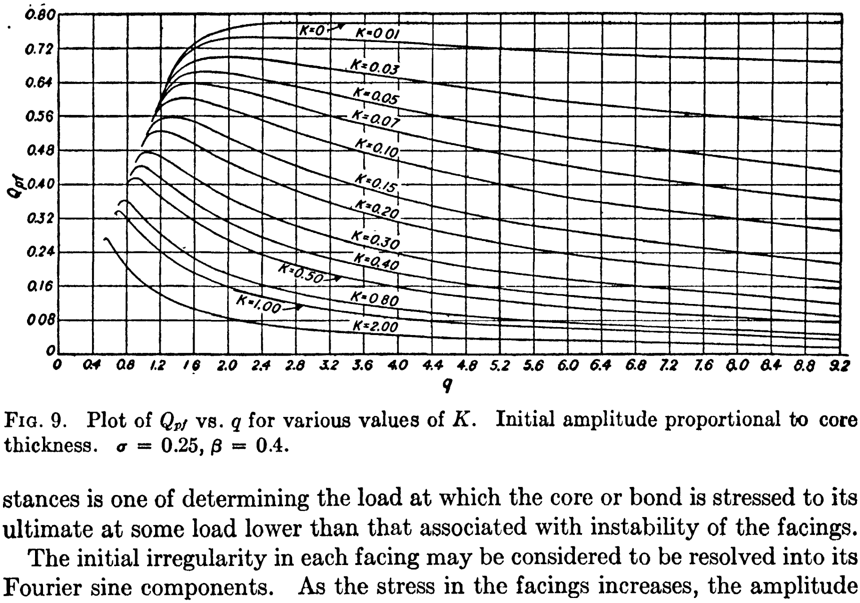

FIG. 9. Plot of Q_{pf} vs. q for various values of K. Initial amplitude proportional to core thickness. $\sigma = 0.25, \beta = 0.4$.

The initial irregularity in each facing may be considered to be resolved into its Fourier sine components. As the stress in the facings increases, the amplitude of each of the Fourier components will change, and one, whose wavelength may be called the ideal wavelength, will ultimately have its amplitude increased out of all proportion to those of the other components. Because there will usually be other Fourier components whose wavelengths differ but little from the ideal, it may very well happen that the combination of the magnified amplitudes of a group of waves of nearly the same wavelength as the ideal will result in a local deformation of the facing from its plane of such magnitude that failure will occur as the result of stresses that exceed the strength of the core or bond.

The effect of initial irregularities is discussed in considerable detail in Forest Products Laboratory Report 1810. Reference will only be made to the results obtained by considering an initial irregularity of a very special type in which the initial form of each facing is expressed by the equation $v_0 = a_0 \sin (\pi x/L)$.

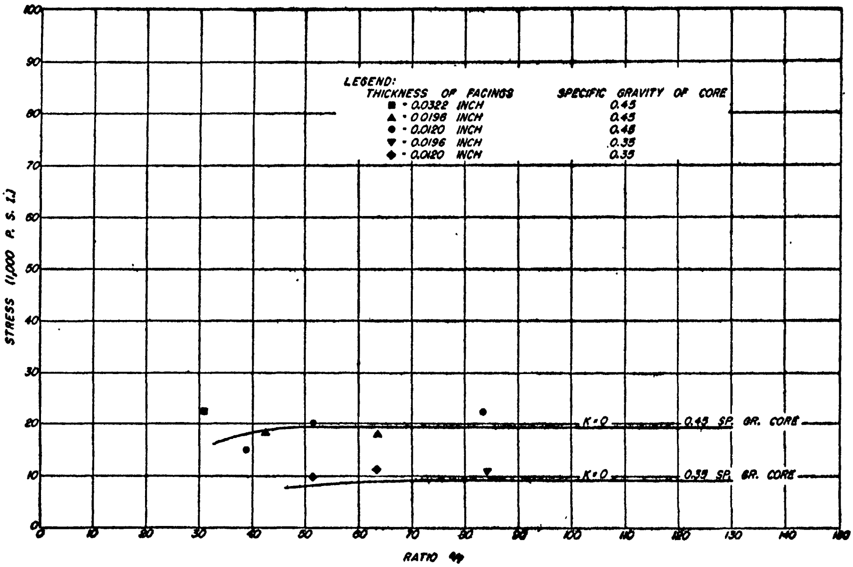

FIG. 10. Sandwich constructions having aluminum facings and cork cores. Curves represent computed values, and each plotted point represents the average of 10 test specimens.

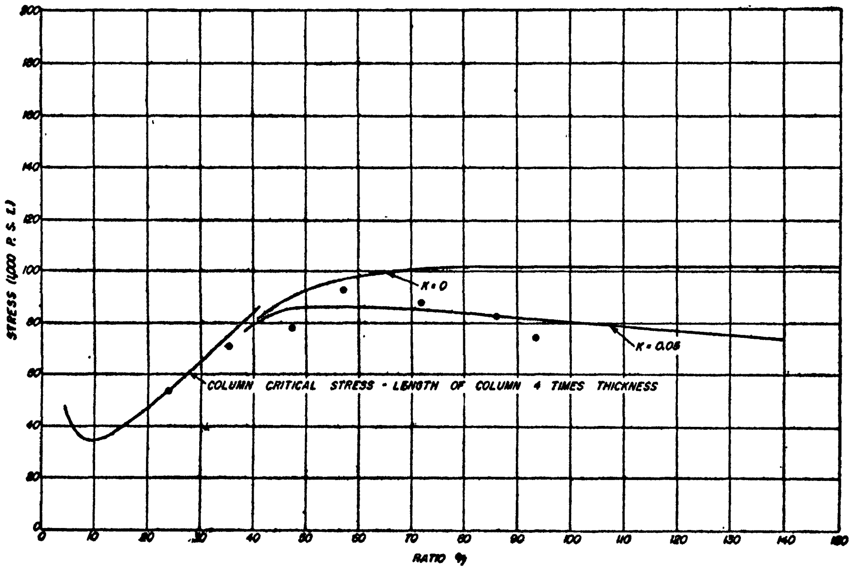

FIG. 11. Sandwich constructions having 0.01-in. steel facings and cellular cellulose acetate cores (flatwise). Curves represent computed values. Each plotted point represents the average of four specimens.

Various assumptions have been made in the literature concerning the amplitude of such an initial wave pattern. Williams [11] assumed that the amplitude was proportional to the wavelength, while Wan [10] assumed that the amplitude was proportional to the square of the wavelength and to the inverse of the face thickness. Both assumptions associate large amplitudes with large wavelengths, and the latter implies that sandwich columns with thick facings are smoother than those with thin facings. As far as the analysis is concerned, one can also assume that waves with amplitude independent of the wavelength are initially present.

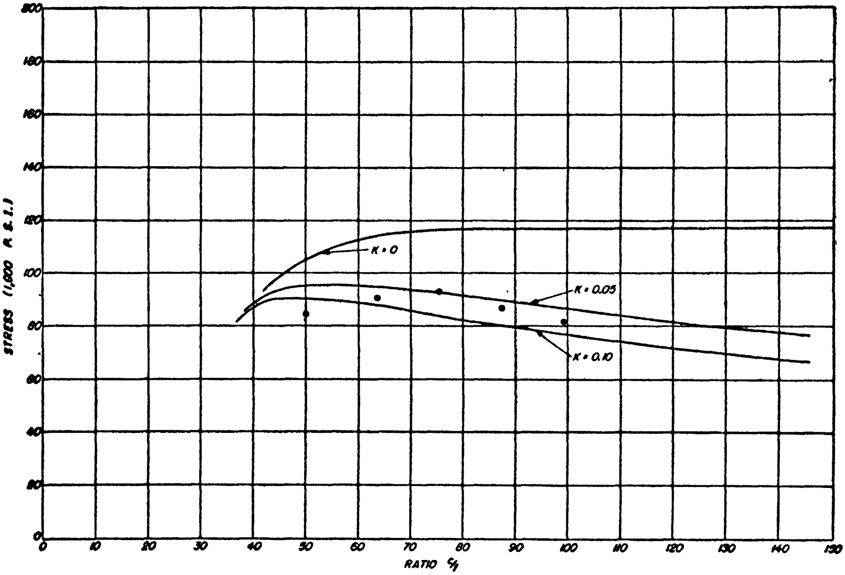

FIG. 12. Sandwich constructions having 0.01-in. steel facings and cellular cellulose acetate cores (edgewise). Curves represent computed values, and each plotted point represents the average of four specimens.

Tests of sandwich columns made at the Forest Products Laboratory indicate that specimens with thick cores and given thickness of facings tend to fail at smaller fractions of the predicted critical load, when assuming the faces to be initially flat, than those with thin cores. Examination of a number of profiles of the specimens gave some indication that those with thick cores were rougher than those with thin cores. It therefore seemed plausible in the analysis of these tests to assume that the original amplitude was proportional to the core thickness for a given thickness of the facings. As to the mode of failure, predictions most consistent with test results were obtained by assuming the wrinkles to be antisymmetrical.

To obtain formulas for predicting the failing stress under the above assumption,

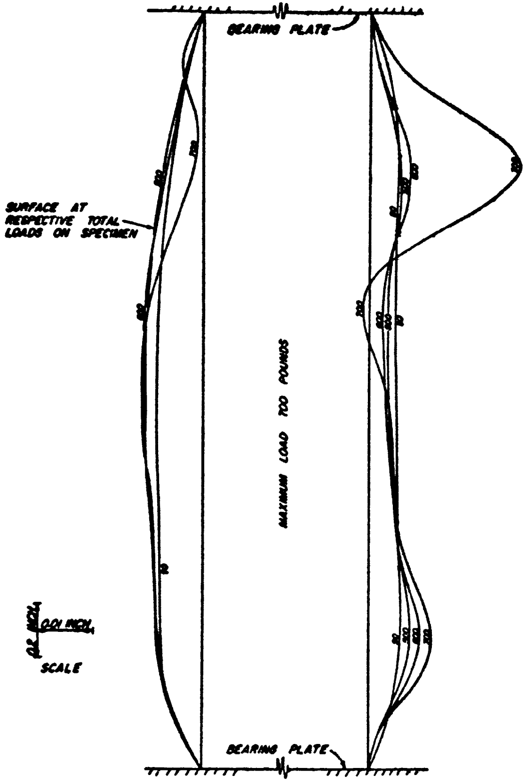

FIG. 13. Profiles of a sandwich construction having 0.012-in. clad aluminum-alloy facings and ¾-in. cork core (0.45 specific gravity). Specimen A.

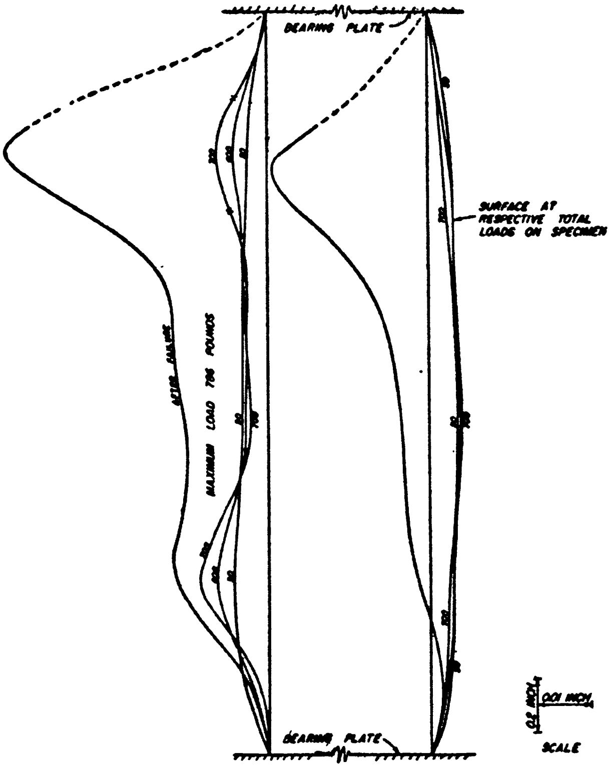

FIG. 14. Profiles of a sandwich construction having 0.012-in. clad aluminum-alloy facings and ¾-in. cork core (0.45 specific gravity). Specimen B.

it is convenient to introduce a dimensionless proportionality factor K and write for the amplitude a_0 of the assumed initial irregularity

$$a_0 = Kca_{22}t,$$

where c is the thickness of the core, a_{22} is defined by (3) or (4), and t is the ultimate value of the stress component Y_y in the core or bond. Curves showing minimum values with respect to L of Q as a function of q are given in Fig. 9 for various values of K. It is to be emphasized that the assumption of the variation of the initial amplitude with the thickness of core, with the thickness of the facings remaining constant, is a tentative one.

12. The predicted wrinkling stress when it is larger than the proportional-limit stress of the face material. When the stress in the facings is above the proportional limit, the modulus of the facing material changes with the stress. In order to predict failure under this condition, the usual procedure, necessarily approximate, is to associate with each stress a reduced modulus E_r. It is not difficult to utilize the foregoing analysis to take this reduced modulus into account. Details of the procedure will be found in Forest Products Laboratory Report 1810.

13. Comparison with the results of tests. Figures 10 to 12 show comparisons of the results of several series of tests with those predicted by theory. In these tests, the stresses in the facings were below the proportional-limit stress of the respective facing materials. It is felt that these comparisons show satisfactory confirmation of the theory. The results of other series of tests are to be found in Forest Products Laboratory Report 1810, together with a discussion of the greater scatter of the experimental points in some of these tests that is taken to be associated with the variability of the elastic properties, of the strength of the core, and of the bond between the facings and the core.

In Figs. 13 and 14 are shown the profiles of the facings of two sandwich columns at successive increments of load. They show the local nature of the wrinkling process and the influence of small initial irregularities.

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DYNAMIC LOADS ON THIN PLATES ON ELASTIC FOUNDATIONS

BY

D. L. HOLL

1. Introduction. An important problem in the safety of structures is that of the induced vibrations in thin plates on elastic foundations caused by impact, pulsating, and moving surface loads. Wheel loads on highway slabs, impact loads of landing airplanes on airport runways, and pulsating loads of heavy industrial machinery with reciprocating parts on floors are examples of this problem.

In the mathematical idealization of this physical problem, it is obvious one cannot introduce all the elements which play a role in the actual model. Several simplifications are made. It is assumed that (*a*) the elementary theory of thin isotropic or anisotropic plates is valid, (*b*) the plate maintains continuous contact with the subgrade and hence the deflection w of the middle surface of the plate is the same as that of its contacting surface, and (*c*) there are no tangential shearing forces at the interface of the plate and its support.

2. The plate equation. The differential equation for the forced vibration of a plate caused by normal surface loads is

$$L(w) + \rho \frac{\partial^2 w}{\partial t^2} = q(x, y; t) - p_s(x, y; t), \tag{1}$$

where $w(x, y; t)$ is the deflection or displacement of the middle surface of the plate, $q(x, y; t)$ is the dynamic load applied normal to the surface, $p_s(x, y; t)$ is the reaction pressure of the elastic base, ρ is the constant mass density per unit of surface area, and $L(w)$ is the fourth-order linear differential operator

$$\begin{aligned} L(w) = D_{11} \frac{\partial^4 w}{\partial x^4} &+ 4D_{16} \frac{\partial^4 w}{\partial x^3\, \partial y} \\ &+ 2(D_{12} + D_{66}) \frac{\partial^4 w}{\partial x^2\, \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x\, \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4}. \end{aligned} \tag{2}$$

This operator expresses the plate's capacity to resist the applied inertial and surface loads. It is valid for a thin anisotropic plate which has the xy plane as a plane of symmetry and where the D's are constants involving the elastic moduli. In an orthotropic plate with two additional planes of elastic symmetry, $D_{16} = D_{26} = D_{66} = 0$. For the isotropic case,

$$L(w) = D\, \nabla^4 w(x, y; t), \tag{3}$$

where $D_{11} = D_{12} = D_{22} = D$. In equation (1) the inertial effect of the applied surface load $q(x, y; t)$ is not considered.

Since the analysis is concerned with the solutions of equation (1) which are valid for the region of the plate, the boundary conditions and initial loading conditions must be known. For plates which are infinite in extent, the deflection w and its derivatives up to the fourth vanish to a suitable order at infinity such that all integrals employed are assumed to exist. In the case of finite rectangular plates treated in Sec. 5, the edges are pinned or simply supported. For the finite circular plate in Sec. 6, the edge has no total moment. It is assumed that initially the plate is at rest when the dynamic loads are applied at $t = 0$,

$$w(x, y; 0) = \frac{\partial w}{\partial t}(x, y; 0) = 0, \tag{4}$$

for all values of x and y except perhaps at the impact point. This implies that $\lim w(x, y; t) = 0$ as $t \to 0^+$, and similarly for the initial velocity.

3. Discussion of reaction forces. From the assumptions (b) and (c) of Sec. 1, the reaction pressure $p_s(x, y; t)$ is normal to the plate surface and is unknown. In this paper three types of reaction are considered. A discussion of these is needed before one attempts solutions of equation (1).

The first type is known as the Winkler [1] hypothesis and treats the reaction pressure as proportional to the local deflection. This case became the basis for several studies made by Hertz [2], Schleicher [3], Happel, [4], Westergaard [5], and other writers. Here the deflection of the plate is also the displacement of the subgrade surface and

$$p_s = kw, \tag{5}$$

where k is a constant known as the subgrade modulus. This implies that the reaction forces may be tensile as well as compressive loads.

In the second type of subgrade, the classical elastic theory of Boussinesq [6] is used for the surface deflection of a semiinfinite deformable medium. If p_0 is a unit load at (ξ, η) on the surface of this elastic half space, the deflection [7] at (x, y) due to p_0 is

$$w(x, y) = k_1 p_0[(x - \xi)^2 + (y - \eta)^2]^{-\frac{1}{2}}, \tag{6}$$

where the constant $k_1 = (1 - \sigma_s)/(\pi E_s)$ with (σ_s, E_s) the moduli of the elastic half space.

A third type of reaction pressure may arise when the deflection of subgrade has the form

$$w(x, y) = p_0 k_2 e^{-k_3[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}, \tag{7}$$

where k_2 and k_3 are constants referring to the subgrade. The types given in equations (6) and (7) belong to a general class of surface "influence functions" represented by

$$w(x, y) = k_s(\,|\,x - \xi\,|\,,\,|\,y - \eta\,|\,), \tag{8}$$

which may be empirically constructed or determined by actual experimental data. These surface influence functions are independent of time, since they represent the surface deflection due to a static load. It is recognized that in an elastic base these surface effects can hardly be independent of time due to the transmission of impulsive wave motion, nevertheless the k_s of equation (8) is assumed to hold in the subsequent analysis.

4. Transform solution for the case of equation (5). We consider the case of an anisotropic plate infinite in extent and acted upon by dynamic loads such that the deflection may be expressed by

$$w(x, y; t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(\alpha x+\beta y)} W(\alpha, \beta; t)\, d\alpha\, d\beta, \tag{9}$$

with

$$W(\alpha, \beta; t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(\alpha x+\beta y)} w(x, y; t)\, dx\, dy, \tag{10}$$

where it is assumed that the integrals exist for $t > 0$. In a similar manner, the normal surface loads $q(x, y; t)$ and $p_s(x, y; t)$ are such that for $|x| \to \infty$, $|y| \to \infty$, their Fourier transforms are

$$Q(\alpha, \beta; t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(\alpha x+\beta y)} q(x, y; t)\, dx\, dy, \tag{11}$$

$$P_s(\alpha, \beta; t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(\alpha x+\beta y)} p_s(x, y; t)\, dx\, dy. \tag{12}$$

Multiplying (1) by $\frac{1}{2\pi} \exp(-i\alpha x - i\beta y)\, dx\, dy$ and integrating over the entire plate, one finds by use of equations (2), (10), (11), and (12) the identity

$$g(\alpha, \beta) W(\alpha, \beta; t) + \rho \frac{\partial^2 W}{\partial t^2} - Q(\alpha, \beta; t) + P_s(\alpha, \beta; t) = 0, \tag{13}$$

where

$$g(\alpha, \beta) = D_{11}\alpha^4 + 4D_{16}\alpha^3\beta + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + 4D_{26}\alpha\beta^3 + D_{22}\beta^4. \tag{14}$$

The polynomial in equation (14) reduces to $D(\alpha^2 + \beta^2)^2$ when the plate is isotropic.

The solution for the transform $W(\alpha, \beta; t)$ in equation (13) can be formally represented when the transform P_s of $p_s(x, y; t)$ is known and the initial conditions which arise from equation (4). In Sec. 7, a study is made of the reaction functions of the type shown in equation (8). In this section, we carry out the solution for the case $p_s = kw$ of equation (5).

Equation (13) becomes

$$\rho W_{tt} + [g(\alpha, \beta) + k] W = Q(\alpha, \beta; t), \tag{15}$$

or

$$W_{tt} + \lambda^4(\alpha, \beta) W = \bar{Q}(\alpha, \beta) \psi(t), \tag{16}$$

where the subscripts denote time derivatives, and $q(x, y; t)$ the applied load is given by $\bar{q}(x, y)\psi(t)$. The constant ρ has been absorbed in the transform $\bar{Q}(\alpha, \beta)$, and

$$\rho\lambda^4 = g(\alpha, \beta) + k. \tag{17}$$

The formal solution for the infinite plate is

$$W(\alpha, \beta; t) = \frac{\bar{Q}(\alpha, \beta)}{\lambda^2} \int_0^t \psi(\tau) \sin \lambda^2(t - \tau)\, d\tau, \tag{18}$$

and

$$\begin{aligned} w(x, y; t) = {} & \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(\alpha x+\beta y)}\, d\alpha\, d\beta \\ & \cdot \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(\alpha\xi+\beta\eta)} \bar{q}(\xi, \eta)\, d\xi\, d\eta \frac{1}{\rho\lambda^2} \int_0^t \psi(\tau) \sin \lambda^2(t - \tau)\, d\tau. \end{aligned} \tag{19}$$

If the applied load does not change its location on the plate, then $\bar{q}(\xi, \eta)$ is known. With $\psi(t)$ given, equation (19) provides the surface deflection.

5. Finite rectangular plate with simply supported edges. We take the geometrical center of the rectangular plate as the origin of the coordinate system and the axes parallel to the edges of the plate which has dimensions $2a$ by $2b$. The conditions at the simply supported rectilinear edges are

$$w(\pm a, y) = w(x, \pm b) = w_{xx}(\pm a, y) = w_{yy}(x, \pm b) = 0. \tag{20}$$

If

$$w(x, y; t) = \sum_{m=1,3,5,\cdots}^{\infty} \sum_{n=1,3,5,\cdots}^{\infty} W_{mn}(t) \cos \alpha x \cos \beta y,$$

with

$$\alpha = \frac{m\pi}{2a}, \qquad \beta = \frac{n\pi}{2b}, \qquad (m, n = 1, 3, 5, \cdots)$$

is inserted in equation (1) for the case employing equation (5), then multiplying by $\cos (r\pi x/2a) \cos (s\pi y/2b)\, dx\, dy$, ($r$, s integers), and integrating over the plate, one finds

$$\begin{aligned} \frac{\partial^2 W_{mn}(t)}{\partial t^2} + [g(\alpha, \beta) + k] \frac{W_{mn}(t)}{\rho} & = q_{mn}(\alpha, \beta; t) \\ & = \frac{1}{4ab\rho} \int_{-a}^{a} \int_{-b}^{b} q(x, y; t) \cos \alpha x \cos \beta y\, dx\, dy. \end{aligned} \tag{21}$$

All the boundary conditions of equation (20) and the initial conditions of equation (4) are satisfied if

$$W_{mn}(\alpha, \beta; t) = \frac{1}{\lambda^2} \int_0^t q_{mn}(\alpha, \beta; \tau) \sin \lambda^2(t - \tau)\, d\tau, \tag{22}$$

where λ is defined in equation (17). The deflection is given by

$$(23)\quad w(x, y; t) = \sum_{(m)} \sum_{(n)} \left[\frac{1}{4ab\rho\lambda^2} \int_0^t \sin \lambda^2(t - \tau)\, d\tau \cdot \int_{-a}^{a} \int_{-b}^{b} q(x, y; \tau) \cos \alpha x \cos \beta y\, dx\, dy \right] \cos \alpha x \cos \beta y.$$

If the plate increases indefinitely such that $a \to \infty$, $b \to \infty$, we can set $h = \pi/2a$, $\alpha = m\pi/2a = mh$, and employ the limiting process

$$\lim_{h \to 0} \sum_{m=1,3,5,\cdots} 2h\, f(mhx) = \int_0^\infty f(zx)\, dz,$$

which converges uniformly in the parameter x. The limit of the double sum in equation (23) becomes

$$(24)\quad w(x, y; t) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \cos \alpha x \cos \beta y\, d\alpha\, d\beta \cdot \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(\xi, \eta; t) \cos \alpha\xi \cos \beta\eta\, d\xi\, d\eta\, \frac{1}{\lambda^2 \rho} \int_0^t \sin \lambda^2(t - \tau)\, d\tau.$$

This agrees with equation (19) when the first double integral is written in exponential form, and $q(x, y; t)$ is of the form $\bar{q}(x, y)\, \psi(t)$.

As an example of the use of equation (23), we can treat the problem of a concentrated load P moving with uniform velocity v across an isotropic finite plate $2a$ by $2b$ along the line $y = y_0$, where $-b < y_0 < b$. It is assumed the load arrives on the plate at time $t = 0$. The moving concentrated load is considered as the limit of a small rectangular load over an area $2cd$ of constant intensity p_0 such that as $c \to 0$, $d \to 0$, $\lim\,(2p_0cd) \to P$. The right member of equation (21) becomes

$$q_{mn}(\alpha, \beta; t) = \lim_{c,d \to 0} \int_{x=vt-c-a}^{x=vt-a} \int_{y=y_0-d}^{y=y_0+d} \frac{p_0 \cos \alpha x \cos \beta y}{4ab\rho}\, dx\, dy = \frac{P}{4ab\rho} \cos \beta y_0 \cos \alpha(vt - a).$$

The final solution in the form of equation (23) is

$$(25)\quad w(x, y; t) = \sum_{m=1,3,5,\cdots}^{\infty} \sum_{n=1,3,5,\cdots}^{\infty} \frac{P}{4\rho ab} \frac{\cos \alpha x \cos \beta y \cos \beta y_0}{\alpha^2 v^2 - \lambda^4} \cdot \left[\cos \lambda^2 t \cos \alpha a + \frac{\alpha v}{\lambda^2} \sin \lambda^2 t \sin \alpha a - \cos \alpha(vt - a) \right].$$

This represents both free and forced vibrations of the plate. A critical condition can be reached when one of the denominator terms in the series becomes equal to zero.

6. Plates with circular symmetry. Consider an infinite plate which is treated as isotropic and the surface load is $q(r, t) = \bar{q}(r)\psi(t)$. Employing equation (5), the plate equation is

$$D\nabla^4 w + \rho w_{tt}(r, t) + kw = q(r, t). \tag{26}$$

Multiply equation (26) by $rJ_0(\xi r)\,dr$ and integrate over the entire domain of the plate. Treating the operator $\nabla^2 w$ in this manner, one has

$$\begin{aligned}
\int_0^\infty (\nabla^2 w) rJ_0(\xi r)\,dr &= \int_0^\infty \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) rJ_0(\xi r)\,dr \\
&= [w_r rJ_0(\xi r)]_0^\infty - \int_0^\infty w_r\,d(rJ_0) + \int_0^\infty w_r J_0(\xi r)\,dr \\
&= -\int_0^\infty w_r[d(rJ_0) - J_0\,dr] = -\int_0^\infty w_r r\xi J_0'(\xi r)\,dr \\
&= -[wr\xi J_0'(\xi r)]_0^\infty + \xi \int_0^\infty w[J_0'(\xi r) + \xi rJ_0''(\xi r)]\,dr \\
&= -\xi^2 \int_0^\infty wrJ_0(\xi r)\,dr = -\xi^2 W(\xi, t),
\end{aligned}$$

since Bessel's function $J_0(\xi r)$ satisfies

$$\xi rJ_0''(\xi r) + J_0'(\xi r) + \xi rJ_0(\xi r) = 0.$$

If $f(r, t)$ is such that as $|r| \to \infty$, the integral

$$F(\xi, t) = \int_0^\infty f(r, t) rJ_0(\xi r)\,dr \tag{27}$$

exists for all values of t, then $F(\xi, t)$ is the Hankel transform of $f(r, t)$. Similar transforms are assumed for $q(r, t)$ and w_{tt}. Repeating the transform of ∇^2, then equation (26) becomes

$$(D\xi^4 + k)W(\xi, t) + \rho W_{tt} = Q(\xi, t),$$

or

$$W_{tt}(\xi, t) + \lambda^4(\xi)W = \frac{Q}{\rho}(\xi, t), \tag{28}$$

where $\rho\lambda^4 = D\xi^4 + k$. Subject to the initial conditions that $w(r, 0) = w_t(r, 0)$, the solution of (28) and its inverse become

$$W(\xi,t) = \frac{1}{\lambda^2\rho}\int_0^t Q(\xi, \tau) \sin \lambda^2(t - \tau)\,d\tau,$$

$$\begin{aligned}
w(r, t) &= H^{-1}\{W(\xi,t)\} = \int_0^\infty W(\xi, t)\xi J_0(\xi r)\,d\xi \\
&= \int_0^\infty \frac{\xi J_0(\xi r)\,d\xi}{\rho\lambda^2} \int_0^t Q(\xi, \tau) \sin \lambda^2(t - \tau)\,d\tau.
\end{aligned} \tag{29}$$

It is not difficult to show that equation (29) can be deduced from equation (19) by considering the properties of circular symmetry.

The multiple infinite integrals of equation (19) may be written

$$\begin{aligned} I &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(\alpha\xi+\beta\eta)} \bar{q}(\xi, \eta)\, d\xi\, d\eta \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(\alpha x+\beta y)} \frac{d\alpha\, d\beta}{\lambda^2(\alpha, \beta)} \\ &= \frac{1}{\pi^2} \int_0^{\infty} \int_0^{2\pi} \bar{q}(R) R\, dR\, d\phi \int_0^{\infty} \int_0^{\infty} \cos\left(\alpha \sqrt{R^2 + r^2 - 2rR\cos\phi}\right) \frac{d\alpha\, d\beta}{\lambda^2(\alpha, \beta)} \\ &= \int_0^{\infty} \bar{q}(R) R\, dR \int_0^{\infty} \frac{\gamma J_0(\gamma R) J_0(\gamma r)\, d\gamma}{\lambda^2(\gamma)} \\ &= \int_0^{\infty} \frac{\bar{Q}(\gamma) \gamma J_0(\gamma r)\, d\gamma}{\lambda^2(\gamma)}. \end{aligned}$$

Then equation (19) becomes

$$w(r, t) = \int_0^{\infty} \frac{\gamma J_0(\gamma r)\, d\gamma}{\rho \lambda^2(\gamma)} \int_0^t Q(\gamma, \tau) \sin \lambda^2(t - \tau)\, d\tau,$$

where $\rho\lambda^4(\gamma) = D\gamma^4 + k$.

The finite circular plate of radius a which has edge conditions $w(a, t) = \nabla^2 w(a, t) = 0$ may be treated by the use of finite transforms. The plate is isotropic, and the subgrade is indicated in equation (26). This equation is multiplied by $rJ_0(\xi_i r)\, dr$ and integrated over the finite domain of the plate. Here the set of parameters ξ_i are chosen to be the positive roots of the transcendental equation

$$(30) \qquad J_0(\xi_i a) = 0.$$

If we denote the finite Hankel transform of $w(r, t)$ by

$$\overline{W}(\xi_i, t) = \int_0^{\infty} r w(r, t) J_0(\xi_i r)\, dr,$$

then from equation (26) one derives an equation similar to (28),

$$\overline{W}_{tt}(\xi_i, t) + \lambda^4(\xi_i)\overline{W} = \frac{Q}{\rho}(\xi_i, t),$$

and all the initial time conditions of equation (4) and the Navier edge conditions are satisfied if

$$\overline{W}(\xi_i, t) = \frac{1}{\lambda^2 \rho} \int_0^t Q(\xi_i, \tau) \sin \lambda^2(t - \tau)\, d\tau.$$

The inverse of this transform yields the solution

$$\begin{aligned} w(r, t) &= \frac{2}{a^2} \sum_{(i)} \frac{\overline{W}(\xi_i, t) J_0(\xi_i r)}{J_1^2(\xi_i a)} \\ &= \frac{2}{a^2} \sum_{(i)} \frac{J_0(\xi_i r)}{\rho \lambda^2 J_1^2(\xi_i a)} \int_0^t Q(\xi_i, \tau) \sin \lambda^2(t - \tau)\, d\tau, \end{aligned}$$

where the summation extends over the positive roots of equation (30), and $\rho\lambda^4(\xi_i) = D\xi_i^4 + k$.

7. General case of elastic subgrade. If the plate maintains continuous contact with its base, then the plate deflection and the surface displacement of the subgrade are the same, and in view of equation (8)

$$(31) \qquad w(x, y; t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_s(\xi, \eta; t) k_s(\mid x - \xi \mid ; \mid y - \eta \mid)\, d\xi\, d\eta,$$

where $k_s(x - \xi, y - \eta)$ is the Green's function of the supporting base.

With the aid of Fourier transforms of the type shown in equations (9) to (12), equation (31) becomes

$$\begin{aligned}(32) \quad W(\alpha, \beta; t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta y)} w(x, y; t)\, dx\, dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta y)}\, dx\, dy \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_s(\xi, \eta; t) k_s(x - \xi, y - \eta)\, d\xi\, d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_s(\xi, \eta; t)\, d\xi\, d\eta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[\alpha(\xi+u)+\beta(\eta+v)]} k_s(u, v)\, du\, dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_s(\xi, \eta; t) e^{i(\alpha\xi+\beta\eta)}\, d\xi\, d\eta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha u+\beta v)}\, k_s(u, v)\, du\, dv \\ &= 2\pi P_s(\alpha, \beta; t) K_s(\alpha, \beta).\end{aligned}$$

The transform of equation (1) becomes

$$(33) \qquad g(\alpha, \beta) W(\alpha, \beta; t) + \rho W_{tt} = Q(\alpha, \beta; t) - P_s(\alpha, \beta; t).$$

The last two equations yield

$$W_{tt} + \left[g(\alpha, \beta) + \frac{1}{2\pi K_s}\right]\frac{W}{\rho} = \frac{Q}{\rho}(\alpha, \beta; t),$$

or

$$(34) \qquad W_{tt} + \lambda^4(\alpha, \beta) W = Q(\alpha, \beta; t), \qquad \rho\lambda^4 = g + \frac{1}{2\pi K_s},$$

and

$$W(\alpha, \beta; t) = \frac{1}{\lambda^2}\int_0^t Q(\alpha, \beta; \tau) \sin \lambda^2 (t - \tau)\, d\tau.$$

The transform $K_s(\alpha, \beta)$ becomes $1/2\pi k$ for equation (5). For the cases given by equations (6) and (7), one has

$$(35) \qquad K_s(\alpha, \beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha u+\beta v)} k_1(u^2 + v^2)^{-\frac{1}{2}}\, du\, dv = \frac{k_1}{(\alpha^2 + \beta^2)^{\frac{1}{2}}},$$

and

$$(36)\quad K_s(\alpha, \beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha u+\beta v)}k_2 e^{-k_3(u^2+v^2)^{\frac{1}{2}}}\,du\,dv = \frac{k_2k_3}{(k_3^2+\alpha^2+\beta^2)^{\frac{3}{2}}}.$$

The deflection becomes

$$(37)\qquad w(x, y; t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta y)}W(\alpha, \beta; t)\,d\alpha\,d\beta.$$

An example employing the transform of equation (35) will be given in the latter part of Sec. 8.

8. Infinite plate with pulsating load. A pulsating load in the form of a uniform pressure is applied over a finite circular area of radius c at time $t = 0$ and has an oscillating time function,

$$(38)\qquad \begin{aligned} q(r, t) &= 0, \qquad r > c \\ &= p_0 \cos \omega t, \qquad r < c, \qquad \omega = \text{const.} \end{aligned}$$

Then

$$Q(\xi, t) = \int_0^c p_0 \cos \omega t\, rJ_0(\xi r)\,dr = \frac{p_0c}{\xi}J_1(\xi c)\cos \omega t,$$

and (29) shows

$$(39)\qquad w(r, t) = \frac{p_0c}{\rho}\int_0^{\infty}\frac{J_0(\xi r)(\cos \lambda^2 t - \cos \omega t)J_1(\xi c)\,d\xi}{(\omega^2 - \lambda^4)}.$$

A special case is the limiting case of a concentrated load $P = \lim (\pi p_0 c^2)$ applied with simple harmonic motion, and

$$(40)\qquad w(r, t) = \frac{P}{2\pi\rho}\int_0^{\infty}\frac{\xi J_0(\xi r)(\cos \lambda^2 t - \cos \omega t)\,d\xi}{\omega^2 - \lambda^4}.$$

Some reduction may be made as follows

$$(41)\quad \begin{aligned} \frac{P\cos \omega t}{2\pi}\int_0^{\infty}\frac{\xi J_0(\xi r)\,d\xi}{D\xi^4 + k - \omega^2} &= \frac{P\cos \omega t}{2\pi DL^2}\int_0^{\infty}\frac{\gamma J_0(r\gamma L)\,d\gamma}{1+\gamma^4} \\ &= \frac{P\cos \omega t}{4DL^2}\left[H_0^{(1)}(rL\sqrt{i}) + H_0^{(2)}(rL\sqrt{-i})\right], \end{aligned}$$

where $DL^4 = k - \rho\omega^2$, $\xi = L\gamma$, and $H_0^{(i)}$ are Hankel's functions with imaginary arguments. This partial result in equation (40) shows that part of the final motion has the same wave form as the deflection of a static load P. The remaining integral does not seem to have a simple closed form.

If equation (35) is utilized for a subgrade transform, the denominator of the first integral in equation (41) has the form $D\xi^4 + k_1\xi - \rho\omega^2$, and no simple closed forms have been found. If $\omega = 0$, that is, if the load is released at $t = 0$

with its full weight, then the resulting first integral

$$\frac{P}{2\pi}\int_0^\infty \frac{J_0(\xi r)\,d\xi}{D\xi^3 + k_1}$$

expresses the actual static deflection caused by a concentrated load P at rest on an infinite plate on such a classical semiinfinite medium. Upon this must be superposed the additional deflection represented by that part of equation (40) which involves the term $\cos \lambda^2 t$.

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KIRCHHOFF'S BOUNDARY CONDITIONS AND THE EDGE EFFECT FOR ELASTIC PLATES

BY

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Suppose a thin plate is subjected to loads and constraints which produce bending. If one edge, or a section of one edge, of the plate is left entirely free, no stresses act against this edge. The approximate solution, however, of the stress problem on the basis of the classical linear theory (the Kirchhoff theory) of the bending of plates yields stresses acting against the free edge which do not all vanish. This should be expected because of the approximate character of the solution; nevertheless one might expect that at least the three resultants—the resultant shear force V, the bending moment M, and the twisting moment H—produced by these stresses would vanish. These latter three conditions were proposed by Poisson in 1828. It was soon realized, however, that this could not be so, because the approximate theory of bending is governed by a differential equation of the fourth order, and hence only two, instead of three, boundary conditions can be imposed. The correct two boundary conditions were formulated by Kirchhoff [1] in 1850. The significance of these conditions was revealed by Kelvin and Tait [2] in 1867 in a manner which is convincing, but not altogether satisfactory.

In 1877, M. Levy [3] proposed a method for refining the classical plate theory which would make it possible to satisfy Poisson's three boundary conditions; Saint-Venant and Boussinesq, however, rejected his proposal. More recently, in 1944, E. Reissner [4] succeeded in modifying the classical bending theory so that Poisson's three conditions could be satisfied; L. Bolle [5] used a somewhat different procedure to reach this result.

The purpose of this present paper is to indicate a systematic derivation of Kirchhoff's boundary conditions and a rational analysis of Kelvin and Tait's argument. This is accomplished by an approach differing from that of any of the authors previously mentioned, namely, by a systematic expansion of the stresses and displacements with respect to powers of the thickness of the plate. The peculiar feature of our approach is the way in which the edge effect is taken into account. In some respects, this procedure is not so handy as that of Reissner, but it has other advantages. In particular, *it enables one not only to satisfy Poisson's three conditions, but in fact to satisfy the complete condition that all three stresses vanish at the free edge. This procedure yields also a justification of Kirchhoff's boundary conditions; furthermore, present results show that the customary treatment of "moderately thick plates"* (see, for example, Love [6]) *by expansion with respect to powers of the thickness is incorrect when applied to plates with a free edge.*

This present paper is in the nature of an abridged preliminary report; at present, investigations are progressing concerning higher order terms and the effect of edge curvature upon the boundary-layer phenomenon. Part of the results presented here have been published earlier [7].

1. Stresses in the interior and at the edge. Let the xy plane be placed at the undeformed middle surface of the plate and denote the plate's thickness by $2c$, so that the plate lies between the planes $z = \pm c$ (see Fig. 1). For the purpose of expansion with respect to the thickness, consider a manifold of plates differing only in thickness. We then expand the values of the quantities in question in powers of c, not at the same point x, y, z but at a point which varies with c so that x, y, $z/c = \zeta$ is fixed.

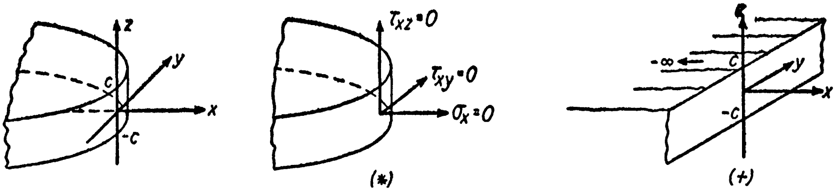

FIG. 1. Elastic plate, free along section of edge.

$$\begin{array}{lll} & \text{Stresses} & \text{Interior} \\ & \sigma_x & = \overset{*}{\sigma}_x + \overset{**}{\sigma}_x + \overset{***}{\sigma}_x + \cdots, \\ & \tau_{xy} & = \overset{*}{\tau}_{xy} + \overset{**}{\tau}_{xy} + \overset{***}{\tau}_{xy} + \cdots, \\ (1) & \sigma_y & = \overset{*}{\sigma}_y + \overset{**}{\sigma}_y + \overset{***}{\sigma}_y + \cdots, \\ & \tau_{xz} & = \qquad\ \overset{**}{\tau}_{xz} + \overset{***}{\tau}_{xz} + \cdots, \\ & \tau_{yz} & = \qquad\ \overset{**}{\tau}_{yz} + \overset{***}{\tau}_{yz} + \cdots, \\ & \sigma_z & = \qquad\qquad\quad \overset{***}{\sigma}_z + \cdots. \end{array}$$

The terms in the expansion so obtained, the "interior stresses," are indicated by asterisks. The number of asterisks indicates the order of the term in c. That means, if n is this number, the term is of the form $f(x, y, \zeta)c^n$. In our notation the factor c^n has been absorbed into the single symbol for each term. Similar expansions are assumed for the "interior displacements."

These expansions of the interior stresses and displacements are inserted into the exact equations of linear elasticity, *i.e.*, the three equilibrium equations and the six equations of Hooke's law. This procedure is exact within the limits of linear elasticity; no extra assumptions, such as a special behavior of the middle surface and of normals to the middle surface, with which the classical theory is encumbered, are here made.

In agreement with the classical approximate theory, the interior expansions of σ_x, τ_{xy}, σ_y; τ_{xz}, τ_{yz}; and σ_z in equations (1) begin with terms of first, second, and third order, respectively. We shall call the interior stresses of lowest order the "Kirchhoff stresses." The terms of next order would then be of third, fourth, and fifth order, respectively, in c if one employed the customary treatment of "moderately thick" plates. Actually however, we show that *the terms of next order which would thus be omitted are different from zero in general; but they can be determined only by analyzing the edge effect.*

The stress resultants V, M, and H are illustrated in Fig. 2.

Consider a segment $x = y = 0$, $|z| \leqq c$, at the free edge, assuming that the x axis is normal and the y axis is tangential there. The exact boundary conditions at this segment are $\sigma_x = \tau_{xy} = \tau_{xz} = 0$. Suppose now that the interior stresses, which are indicated by asterisks in equations (1) and for which the independent variables are x, y, z/c, were known by obtaining them from an expansion as described. If one evaluates these interior stresses at the edge, they do not, as is known, satisfy the three correct boundary conditions. The reason is that the expansions indicated are not uniform in the neighborhood of the edge and are not valid at the edge. (Incidentally, these expansions cannot be expected to be convergent; they should be assumed to be asymptotic in character; but this fact is irrelevant in the present context.) The actual stresses vary quickly upon approaching the edge; the more quickly, the smaller the thickness. The quickly varying stresses which are to be superimposed on the interior stresses (so that the sum will give the actual stresses both in the interior and at the edge) will be called the "excess stresses," indicated by + superscripts.

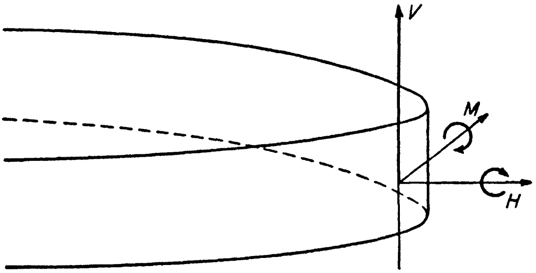

FIG. 2. Moments and shear force at the edge (per unit length).

Stresses Interior Excess

$$
\begin{aligned}
\sigma_x &= \overset{*}{\sigma}_x + \overset{**}{\sigma}_x + \overset{***}{\sigma}_x + \cdots + \overset{+}{\sigma}_x + \overset{++}{\sigma}_x + \cdots,\\
\tau_{xy} &= \overset{*}{\tau}_{xy} + \overset{**}{\tau}_{xy} + \overset{***}{\tau}_{xy} + \cdots + \overset{+}{\tau}_{xy} + \overset{++}{\tau}_{xy} + \cdots,\\
\sigma_y &= \overset{*}{\sigma}_y + \overset{**}{\sigma}_y + \overset{***}{\sigma}_y + \cdots + \overset{+}{\sigma}_y + \overset{++}{\sigma}_y + \cdots,\\
\tau_{xz} &= \overset{**}{\tau}_{xz} + \overset{***}{\tau}_{xz} + \cdots + \overset{+}{\tau}_{xz} + \overset{++}{\tau}_{xz} + \cdots,\\
\tau_{yz} &= \overset{**}{\tau}_{yz} + \overset{***}{\tau}_{yz} + \cdots + \overset{+}{\tau}_{yz} + \overset{++}{\tau}_{yz} + \cdots,\\
\sigma_z &= \overset{***}{\sigma}_z + \cdots + \overset{++}{\sigma}_z + \cdots.
\end{aligned}
\tag{1^+}
$$

These expansions will now be valid both in the interior and at the edge.

In order to obtain the excess stresses, a different expansion process is needed; in it the point at which the stresses are expanded varies with the thickness so that x/c, y/c, z/c, is fixed. Figure 3 shows three typical cases of the behavior of the interior and excess stresses.

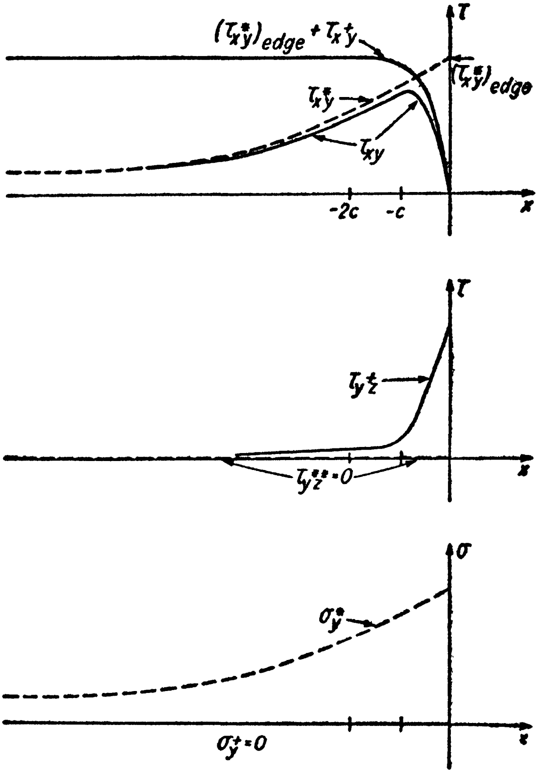

FIG. 3. Fitting together of Kirchhoff stresses and excess stresses.

We proceed to show that the two types of terms in the full expansions (1^+) cannot be determined independently of each other. In the successive steps, one is required to use interior stresses in order to determine excess stresses and vice versa. The links between these two types of stresses are the *boundary conditions*:

$$\begin{array}{cc} \text{At the edge} & \text{At } x = -\infty \\ \left\{\begin{matrix} \overset{*}{\sigma}_x \\ \overset{*}{\tau}_{xy} \end{matrix}\right\} + \left\{\begin{matrix} \overset{+}{\sigma}_x \\ \overset{+}{\tau}_{xy} \\ \overset{+}{\tau}_{xz} \end{matrix}\right\} = 0, & \left\{\begin{matrix} \overset{+}{\sigma}_x \\ \overset{+}{\tau}_{xy} \\ \overset{+}{\tau}_{xz} \end{matrix}\right\} = 0, \\ \left\{\begin{matrix} \overset{**}{\sigma}_x \\ \overset{**}{\tau}_{xy} \\ \overset{**}{\tau}_{xz} \end{matrix}\right\} + \left\{\begin{matrix} \overset{++}{\sigma}_x \\ \overset{++}{\tau}_{xy} \\ \overset{++}{\tau}_{xz} \end{matrix}\right\} = 0, & \left\{\begin{matrix} \overset{++}{\sigma}_x \\ \overset{++}{\tau}_{xy} \\ \overset{++}{\tau}_{xz} \end{matrix}\right\} = 0, \end{array}$$

$$\text{(2)} \qquad \text{At the edge}$$

$$\frac{\partial}{\partial y}\overset{*}{\sigma}_x + \frac{\partial}{\partial y}\overset{++}{\sigma}_x = 0,$$

$$\frac{\partial}{\partial y}\overset{*}{\tau}_{xy} + \frac{\partial}{\partial y}\overset{++}{\tau}_{xy} + \kappa\overset{*}{\sigma}_y = 0,$$

$$\frac{\partial}{\partial y}\overset{++}{\tau}_{xz} + \kappa\overset{+}{\tau}_{yz} = 0.$$

Clearly the sums of the interior and excess stress contributions to σ_x, τ_{xy}; and τ_{xz} vanish at the free edge.

This latter group of equations results from the fact that the rate of change of the sum of the interior and excess stresses along the edge must be equal to zero.

The excess stresses are small at a distance from the edge which depends upon the thickness $2c$ and is roughly proportional to it. It can be shown that, in the limit case of zero thickness, the precise condition is that the excess stresses should approach zero as $\xi = x/c$ approaches $-\infty$. It is also necessary to formulate the fact that the rate of change of these stresses along the edge is zero; in this way the curvature κ of the edge enters. For simplicity, however, we shall assume the curvature to be zero in the following discussion.

2. Interior and excess stresses and the first Kirchhoff condition. Before describing the results in detail, we first explain how the excess stresses and the Kirchhoff stresses are used for a description of the complete stress system. Consider, for example, the stress τ_{xy} along a normal to the edge at the specific edge point $x = y = 0$. The stress should vanish at the segment $|z| \leqq c$ through this point. In general, the interior stress derived from the Kirchhoff theory does not vanish there as, indeed, it approaches a nonzero value at the edge. If, however, one superimposes the excess stress on it, an approximation to the actual stress is obtained which is correct up to the edge (within terms of higher order in c than those used). It is interesting to remark that it may happen that the excess stress is of higher order than the Kirchhoff stress; this occurs, for example, for the stress σ_y because, as will be shown later, the first excess term $\overset{+}{\sigma}_y$ actually vanishes. Or it may happen, vice versa, as for τ_{yz}, that the interior stress is of higher order than the excess stress. In this latter case, we see that the Kirchhoff theory leads to a definite underestimation of the stress.

The excess stress of the first order consists of a purely torsional stress system about the tangent as axis plus a possible plane strain system.

$$\text{(3)} \qquad \{\overset{+}{\sigma}_x, \cdots, \overset{+}{\sigma}_z\} = \underset{\text{Torsion}}{\{\overset{+}{\tau}_{xy}, \overset{+}{\tau}_{yz}\}} + \underset{\text{Plane strain}}{\{\overset{+}{\sigma}_x, \overset{+}{\sigma}_y, \overset{+}{\tau}_{xz}, \overset{+}{\sigma}_z\}}.$$

The torsional system differs from the stress systems which occur in the torsion of a bar in that it involves a twisting moment at the edge but no twisting deformation. The value of this twisting moment can be determined only by subsequent steps of our procedure.

The first terms of the stress resultants, both from the interior and the excess stresses, are defined below. For these resultants, the number of asterisks or plus signs refers, not to the order in c, but rather to the order of the stresses which produce these resultants.

Bending moment:

$$M^* = \int_{-c}^{c} \overset{*}{\sigma}_x z \, dz, \qquad M^+ = \int_{-c}^{c} \overset{+}{\sigma}_x z \, dz,$$

Twisting moment:

$$H^* = \int_{-c}^{c} \overset{*}{\tau}_{xy} z \, dz, \qquad H^+ = \int_{-c}^{c} \overset{+}{\tau}_{xy} z \, dz, \tag{4}$$

Shear force:

$$V^{**} = \int_{-c}^{c} \overset{**}{\tau}_{xz} \, dz, \qquad V^+ = \int_{-c}^{c} \overset{+}{\tau}_{xz} \, dz.$$

We note that the three Poisson conditions are $M^* = H^* = V^{**} = 0$, whereas the two Kirchhoff conditions are

$$M^* = 0, \qquad V^{**} + \frac{d}{ds} H^* = 0. \tag{5}$$

A second possible contribution to the excess stress system of first order could be a plane strain system. It can be shown that such a system would possess no shear force V and no bending moment M. Hence the complete excess stresses of first order have no bending moment at the edge, *i.e.*,

$$M^+ = 0. \tag{6}$$

One can, in fact, prove that the plane strain system vanishes identically because of the boundary condition. The arguments which give this result at the same time yield the relation $M^* = -M^+$, hence

$$M^* = 0. \tag{7}$$

Thus, the *first Kirchhoff boundary condition is justified.*

3. The second Kirchhoff condition. In order to justify the second Kirchhoff boundary condition, it is first necessary to describe the excess stresses of the second order. It can be shown that these stresses consist of a plane strain system, which does not vanish this time, plus a stress system which one may call an "integrated torsional stress system." This means that the derivatives of this latter stress system with respect to y form a torsional stress system of the type described previously.

$$\{\sigma_x^{++}, \cdots, \sigma_z^{++}\} = \underset{\textbf{Integrated torsion}}{\{\sigma_x^{++}, \cdots, \tau_{yz}^{++}\}} + \underset{\textbf{Plane strain}}{\{\sigma_x^{(p)++}, \cdots, \sigma_z^{++}\}}, \tag{8}$$

where

$$\frac{\partial}{\partial y}\{\sigma_x^{++}, \cdots, \sigma_z^{++}\} = \underset{\textbf{Torsion}}{\left\{\frac{\partial}{\partial y}\tau_{xy}^{++}, \frac{\partial}{\partial y}\tau_{yz}^{++}\right\}}.$$

For this integrated torsional system, one can derive the relation

$$V^{++} + \frac{d}{dy} H^{++} = 0. \tag{9}$$

This can be done in a simple and rigorous manner which would seem to be more satisfactory than the argument of Kelvin and Tait. By means of those boundary conditions (2), which connect the excess stresses τ_{xy}^{++}, τ_{xz}^{++} with the Kirchhoff stresses τ_{xy}^{*}, τ_{xz}^{**}, one then derives from condition (9) that the *second Kirchhoff boundary condition holds*:

$$V^{**} + \frac{d}{dy} H^{*} = 0. \tag{10}$$

Only at this stage of the procedure are we finally in a position to determine definitely the Kirchhoff stresses σ_x^*, τ_{xy}^*, σ_y^*, τ_{xz}^{**}, τ_{yz}^{**}, σ_z^{***}, and also the resultants V^{**} and H^*. As a consequence, the excess quantities of lowest order $H^+ = -H^*$, τ_{xy}^+ and τ_{yz}^+ are now completely determined. In particular, for the stress τ_{yz} at the top of the edge ($x = 0$, $z = c$), we obtain

$$\begin{aligned} \tau_{yz}^{**} &= 0, \qquad \text{and} \\ \tau_{yz}^{+} &= -1.11c^{-2}H^*, \end{aligned} \tag{11}$$

which are of the same order of magnitude as σ_y^* at $x = 0, z = 0$. These equations result from an actual calculation of the quantities involved in terms of known quantities.

4. Quantities of higher order. One can of course continue the procedure outlined to determine terms of higher order of the interior and excess stresses. In the present stage, the excess stresses of second order are not yet completely known; nevertheless, the integrated torsion system would already be determined within an additional torsion if $V^{++} = -V^{**}$ were known. In particular, the moment M^{++} can be expressed in terms of V^{++}, and a numerical calculation yields

$$M^{++} = -.63V^{++} = .63V^{**}, \tag{12}$$

and thus M^{**} is found to be

$$M^{**} = -M^{++} = -.63V^{**}. \tag{13}$$

This moment M^{**} is in general different from zero. Consequently *the terms of next order of the interior stresses, σ_x^{**} and the others, are not all zero. This fact shows that the procedure employed in the theory of moderately thick plates, in which σ_x^{***} and so on are taken as the terms of next order, is not correct when used for plates with a free edge.*

As justification for this procedure, the argument is frequently offered that in actual problems one does not know precisely how the stresses are distributed at an edge, and that therefore one can be content with satisfying merely the three Poisson boundary conditions on the resultants; but if this is done, then one should be consistent and not seek for higher order terms in the solution, beyond those of the Kirchhoff theory, which depend on a more accurate knowledge of the boundary data than is available. In the case of a free edge considered here, the excuse does not apply anyway; for, here it is known exactly how the stresses are distributed: they are exactly zero.

Our procedure clearly can be extended to obtain a refinement of the description of the interior stresses, including curvature effects by the methods indicated. The intention is to present details in a later publication.

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SOME PROBLEMS IN THE BENDING OF THIN PLATES

BY

G. F. CARRIER AND F. S. SHAW

1. Introduction. We shall consider a class of problems associated with the bending of thin plates (or with other physical problems) having the following characteristics. In an appropriate coordinate system, frequently the physical plane, the boundary of the region under consideration consists of two radial lines through the origin and some curve closing the contour so that the region is simply connected. The boundary conditions on the radial lines are homogeneous, but those on the remainder of the boundary are not necessarily so. The solution of the linear homogeneous differential equation associated with the problem is then represented by a linear combination of eigenfunctions or "corner functions" each of which obeys the differential equation and the boundary conditions on the radial edges. We do not show that an infinite series of these eigenfunctions will rigorously converge to the solution of the problem but, in the cases where it has been used, we have found that a very small number of terms will describe the state of affairs accurately enough for practical purposes. We present our results in the belief that this type of representation will frequently require less labor than will various other well-known techniques.

2. Some typical problems. It would appear most efficient to illustrate the representation mentioned above by some simple specific examples. For example, let the circular edge ($r = 1$) of a sector-shaped plate be clamped and let a unit torque be applied at, or very near, the vertex. The radial edges should be free of stress. The differential equations for the deflection $w(r, \theta)$ become, according to the classical small bending theory [1],

$$\Delta\Delta w = 0, \tag{1}$$

where Δ is the Laplacian operator. The boundary conditions on the radial edges are

$$M_\theta(r, \alpha) = M_\theta(r, -\alpha) = V_\theta(r, \alpha) = V_\theta(r, -\alpha) = 0, \tag{2}$$

where

$$M_\theta(r, \theta) = -D\left[\frac{w_r}{r} + \frac{w_{\theta\theta}}{r^2} + \nu w_{rr}\right],$$

$$V_\theta(r, \theta) = -D\left[\frac{\Delta(w_\theta)}{r} + (1 - \nu)\left(\frac{w}{r}\right)_{rr\theta}\right],$$

and where $\theta = \pm\,\alpha$ defines the radial edge of the plate. On the circular edge, we have

$$w(1, \theta) = w_r(1, \theta) = 0. \tag{3}$$

We also require that

$$\int_{-\alpha}^{\alpha} [r^2 V_r(r, \theta) \sin \theta + r M_r(r, \theta) \cos \theta] \, d\theta = 1. \tag{4}$$

A fundamental solution [2] of equation (1) which satisfies (2) and (4) is

$$w_0(r, \theta) = C \left[r \ln r \sin \theta + \frac{1 + r}{2} r\theta \cos \theta \right], \tag{5}$$

with $2/C = D(1 - \nu)^2(2\alpha - \sin 2\alpha)$.

However, this function fails to satisfy the condition at $r = 1$. Thus, we must superimpose a set of eigenfunctions which represent zero torque but modify $w(1, \theta)$ and $w_r(1, \theta)$. Such eigenfunctions are easily found in the form

$$w_j(r, \theta) = r^{n_j} f_j(\theta). \tag{6}$$

The f_j must, in fact, be of the form $f_j = \sin (n_j\theta) + b_j \sin [(n_j - 2)\theta]$. The boundary conditions at $\theta = \pm\alpha$ lead to the characteristic equation

$$2\alpha(3 + \nu) \sin z_j + (1 - \nu) z_j \sin 2\alpha = 0, \tag{7}$$

where

$$z_j = 2(n_j - 1)\alpha; \tag{8}$$

also,

$$b_j = \frac{n_j(1 - \nu) \sin n_j\alpha}{[4 - n_j(1 - \nu)] \sin [(n_j - 2)\alpha]}.$$

We must not omit the solution corresponding to a rigid body rotation of the plate, *i.e.*,

$$w^*(r, \theta) = r \sin \theta. \tag{9}$$

This, of course, corresponds to the solution $z_j = 0$ of equation (7).

Let us now consider the case $\alpha = \pi/12$, $\nu = 0.3$. The first few roots for which $\operatorname{Re}(n_j) > 0$ are, respectively, $n_1 = 8.83$, $n_2 = 10.37$, $n_{3,4} = 21.72 \pm 2.75i, \cdots$. However, we can actually find an excellent approximation to the solution of this problem if we utilize only w^* and two of the nontrivial eigenfunctions, *i.e.*, only the real ones.

We write

$$w \simeq w_0 + a^* w^* + a_1 w_1 + a_2 w_2. \tag{10}$$

The most accurate method for choosing the a_j might be to minimize $\int_{-\alpha}^{\alpha} [w^2(1, \theta) + N w_r^2(1, \theta)] \, d\theta$, but a rapid method is to write

$$\int_0^{\alpha} w(1, \theta) \, d\theta = \int_0^{\alpha} w_r(1, \theta) \, d\theta = 0,$$

and for the third condition (on the three a_j) we can demand that $w(1, \alpha)$ vanish. When the foregoing integrations are performed, we obtain three linear algebraic equations for a^*, a_1, a_2, and the solution of these leads to the result

$$w = w_0 - C(0.516w^* + 0.0226w_1 + 0.0009w_2). \tag{11}$$

In order to estimate the accuracy with which this solution obeys the boundary conditions at $r = 1$, it is convenient to define as a reference quantity $W = w_0 - 0.516Cw^*$. That is, we shall compare the amount by which $w(1, \theta)$, $w_r(1, \theta)$ fail to vanish with the amount by which the fundamental solution corrected by a rigid body rotation fails to vanish at this edge. Using this "normalized" error, we find that the mean square values of $w(1, \theta)/W(1, \theta)$ and of $w_r(1, \theta)/W_r(1, \theta)$ are about 0.01. Thus the solution given by equation (11) is sufficiently accurate for most purposes.

A second problem whose solution by conventional means once took 3 months of computation [3] is the following: Consider a clamped sectorial plate under a uniform load. Then

$$\Delta\Delta w = \frac{p}{D} = 192\,P \tag{12}$$

and $w = \partial w/\partial n = 0$ on $\theta = \pm\alpha$, and on $r = 1$. The solution of this problem can be represented by a nonhomogeneous solution of equation (12) for which $w(r, \pm\alpha) = w_\theta(r, \pm\alpha) = 0$ plus a series of biharmonic eigenfunctions obeying these same conditions on the radial edges. For concreteness, we take $\alpha = \pi/6$, and the nonhomogeneous solution becomes

$$w_0 = Pr^4(3 + 2\cos 4\theta - 4\cos 2\theta), \tag{13}$$

and the eigenfunctions are again of the form $w_j = r^{n_j}f_j(\theta)$. This time the characteristic equation associated with $w = w_\theta = 0$ on $\theta = \pm\alpha$ is

$$2\alpha \sin z_j + z_j \sin 2\alpha = 0 \tag{14}$$

with z_j again given by

$$z_j = 2(n_j - 1).$$

Since w_0 and the eigenfunctions each obey the boundary conditions at $\theta = \pm\alpha$, we have merely to fix the condition at $r = 1$. This time we write

$$w = w_0 + \sum D_n w_n + \bar{D}_n \bar{w}_n \tag{15}$$

and orthogonalize both w and w_r at $r = 1$ against $e^{ik\theta\pi/\alpha}$ for enough integers k so that the a_j are determined.[1] That is, we write $\int_0^\alpha w(1, \theta)e^{ik\pi\theta/\alpha}\,d\theta = 0$, etc.

Actually, in this problem, we need merely consider the function w_1 and its conjugate and utilize only $k = 0$. When this is done, we find, since $n_1 = 5.059 + 1.952\, i$,

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[1] This method of approximating the boundary condition was first heard by the authors at a lecture by A. Weinstein.

$$D_1 = P(0.384 - 0.276\, i). \tag{16}$$

When this is utilized in equation (15), we find that the mean square error in $w(1, \theta)$ (as defined in the previous problem) is well under 0.01 and that in w_r is about 0.02.

3. General remarks. (1) It is not at all necessary that the eigenfunctions to be used in this manner should be of the simple product type. However, in order that the method constitute a quick route to a result, it is essential that the eigenfunctions be easily computed. (2) The ease with which the integrations associated with the satisfaction of the (usually) nonhomogeneous part of the boundary condition can be carried out is not very important. A few one-dimensional numerical integrations are very quickly performed. If, in the second problem, for example, a wavy edge were chosen to replace the circular one, the integrations would be nasty analytically but simple numerically, and the solution would still be readily available. (3) It cannot be anticipated for highly oscillatory boundary curves or boundary conditions that the solution can be accurately given in a very few terms. For smooth edge conditions, however, it seems true that a reasonably compact result with negligible inaccuracies can be found.

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PRESTRESSING A PLANE CIRCULAR PLATE TO STIFFEN IT AGAINST BUCKLING[1]

BY

J. J. STOKER

A thin plane elastic plate subjected to gradually increasing normal pressure at its edges and in its plane will eventually become unstable and buckle or bend when the pressure reaches a certain critical value. Our principal object is to discuss a means by which the critical buckling load in compression for the clamped circular plate (and, by inference, for plates with other shapes and boundary conditions as well) can be increased through prestressing the plate in the following manner: *A slit is cut in the plate, a wedge of small angle α is inserted in such a way as to open out the plate fanwise, and the wedge is then welded to the plate along its edges. If the plate is then subjected to additional compressive stresses in its plane, it will not buckle until these stresses are larger than they would be for buckling of the initially unstressed plate.* The lowest critical buckling pressure is increased by increasing the wedge angle α up to a certain point, but it is a striking and peculiar fact that *there is an upper critical limit α_c for the wedge angle α at which the lowest critical compressive stress attains a maximum*; any further increase in α beyond α_c lowers the lowest critical pressure. However, the lowest critical buckling pressure can be tripled by choosing the critical value α_c for the wedge angle α.

This rather novel phenomenon can be understood physically as follows: If the lowest critical pressure were always associated with a radially symmetric buckling mode, it would be possible to increase it indefinitely by increasing the wedge angle. However, it turns out as a result of our calculations that the lowest critical pressure is furnished by a nonsymmetrical buckling mode with several nodal diameters once the angle α exceeds a certain value, and the critical pressures for these buckling modes *decrease* with increase of α beyond the critical value defined above. This latter occurrence may in turn be understood from the fact, also determined by our calculations, that insertion of a wedge of sufficiently large angle induces initial stresses which make the plate unstable even though the pressure at the boundary is zero. This occurs, however, only in buckling modes with several nodal diameters.

From the mathematical point of view, the problem is a linear eigenvalue problem containing two parameters (the wedge angle α and the critical buckling pressure) which is associated with the biharmonic equation.

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[1] Abstract. A short version of this paper is published under the title *Pre-stressing a plane circular plate to stiffen it against buckling*, in the Reissner Anniversary Volume, Contributions to Applied Mechanics, J. W. Edwards, Ann Arbor, Michigan, 1949, pp. 268–276.

THE BENDING OF THIN ELLIPTIC PLATES[1]

BY

C. L. PERRY

1. Introduction. The purpose of this paper is to present two series solutions to certain elliptic plate problems. The plates considered are assumed to be elastically isotropic and homogeneous, and of uniform thickness. A thin plate is any plate of which the thickness $2h$ is much smaller than the smallest lateral dimension of the plate. The bending of the plate is due to a static loading which is perpendicular to the faces of the plate and to the supports at the edge of the plate. When the elliptic boundary of the plate reduces to a circle, one of the series solutions (a Mathieu function solution) reduces to the Fourier-Bessel and Fourier-Dini solutions for the circular plate. Solutions of this type have been given by S. Jen [1]. The other solution (a series of hyperbolic and trigonometric functions) reduces to a Fourier-power series solution for the circular plate [2].

The coordinate systems used to describe the position of the plate are determined by the plate in its unstrained state. Let us take, as the origin of a rectangular coordinate system (x_0, x_1, x_2), the middle point of the plate. Let the minor axis of the ellipse in the middle plane of the plate be the x_2 axis and the unit of length be the length of the semiminor axis of this ellipse. Let the major axis of this ellipse be the x_1 axis and the distance normal to the x_1, x_2 plane be the third coordinate x_0. In order that the conditions given on the elliptical boundary of the plate may be described as functions of only one coordinate, it is advantageous to introduce elliptic cylinder coordinates (ξ_0, ξ_1, ξ_2). These coordinates are related to the rectangular coordinates by the following system of equations:

$$\begin{aligned} x_0 &= \xi_0, \\ x_1 &= \mathrm{d} \cosh \xi_1 \cos \xi_2, \\ x_2 &= \mathrm{d} \sinh \xi_1 \sin \xi_2, \end{aligned} \tag{1}$$

where $2d$ is the interfocal distance of the elliptical boundary. The parametric lines ξ_1 equal to a constant form a family of confocal ellipses in the x_1, x_2 plane with foci at $x_1 = \pm d$; $x_2 = 0$. The parametric lines ξ_2 equal to a constant form a family of hyperbolas that are orthogonal to the ellipses and have the same foci. The equation of the elliptical boundary of the plate is

$$\xi_1 = \sinh^{-1} \frac{1}{d} = \bar{\xi}, \qquad -h \leq \xi_0 \leq h. \tag{2}$$

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[1] This paper represents part of the work carried out under an Atomic Energy Commission Fellowship at the University of Michigan under the supervision of Professor G. E. Hay.

The relation between the rectangular coordinates x_1, x_2 and the elliptical coordinates ξ_1, ξ_2 can be studied by the conformal mapping from a complex plane $\xi_1 + i\xi_2$ to the complex plane $x_1 + ix_2$. The mapping is given by $x_1 + ix_2 = d \cosh (\xi_1 + i\xi_2)$. The region $0 \leqq \xi_1 \leqq \bar{\xi}$, $-\pi < \xi_2 \leqq \pi$ maps on the entire ellipse. The critical points of the mapping are $\xi_1 = 0, \xi_2 = 0$; $\xi_1 = 0$, $\xi_2 = \pi$. These points map into the foci $x_1 = \pm d$, $x_2 = 0$ of the elliptical boundary. The segment $-d \leqq x_1 \leqq d$, $x_2 = 0$ is covered twice. For a function $F(\xi_1, \xi_2)$ to be continuous in the interior of the ellipse, F must be continuous in ξ_1 and ξ_2 $(0 \leqq \xi_1 \leqq \bar{\xi}, -\pi < \xi_2 \leqq \pi)$ and also satisfy the conditions:

$$\lim_{\xi_1 \to 0} F(\xi_1, \xi_2) = \lim_{\xi_1 \to 0} F(\xi_1, -\xi_2), \qquad (0 \leqq \xi_1 \leqq \bar{\xi}, -\pi < \xi_2 \leqq \pi) \tag{3}$$

$$\lim_{\xi_2 \to \pi} F(\xi_1, \xi_2) = \lim_{\xi_2 \to -\pi} F(\xi_1, \xi_2). \qquad (0 \leqq \xi_1 \leqq \bar{\xi}, -\pi < \xi_2 \leqq \pi) \tag{4}$$

All the quantities that we consider are assumed to have been rendered dimensionless by division by appropriate powers of a standard length L (chosen above as the length of the semiminor axis of the elliptical boundary of the plate), and Young's modulus E. The nondimensional thickness $2h \ll 1$ (for thin plates).

2. Differential equations for deflection and stress. In a recent series of four papers, Synge and Chien [3, 4] systematized the various approximation theories for thin plates. Some of Chien's results yield the classical differential equations of Lagrange and von Kármán, and in addition show the restriction on the magnitude of the surface loading and edge loading for the terms omitted from the differential equation to be small. Further, his equations are for a general curvilinear coordinate system and are thus easily specialized to elliptic cylinder coordinates. We shall now express Lagrange plate equations for deflections of higher order than the thickness of the plate in elliptical coordinates and state the restrictions on the magnitude of the loadings:

$$\Delta\Delta w(\xi_1, \xi_2) = -\frac{1}{D} q(\xi_1, \xi_2), \tag{5}$$

$$\Delta\Delta F(\xi_1, \xi_2) = \frac{2h}{\det (g_{\alpha\beta})} (w_{|12} w_{|12} - w_{|11} w_{|22}), \tag{6}$$

where w is the deflection (change in ξ_0) of the points in the middle plane of the plate, F is a function whose derivatives yield the membrane stresses, $2h$ is the plate thickness, Δ is the Laplacian operator

$$\Delta = \frac{2\Delta^*}{C_2 - c_2}, \qquad \Delta^* = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_1^2}, \qquad C_2 = \cosh 2\xi_1, \qquad c_2 = \cos 2\xi_2,$$

D is the flexural rigidity $h^3/(1 - \sigma^2)$ of the plate, and the covariant derivative operators, $_{|11}$, $_{|12}$, $_{|22}$ are given by the following:

$$w_{|11} = \frac{\partial^2 w}{\partial \xi_1^2} - \frac{S_2}{C_2 - c_2} \frac{\partial w}{\partial \xi_1} + \frac{s_2}{C_2 - c_2} \frac{\partial w}{\partial \xi_2}, \tag{7}$$

(8) $$w_{|22} = \frac{\partial^2 w}{\partial \xi_2^2} + \frac{S_2}{C_2 - c_2}\frac{\partial w}{\partial \xi_1} - \frac{s_2}{C_2 - c_2}\frac{\partial w}{\partial \xi_2},$$

(9) $$w_{|12} = \frac{\partial^2 w}{\partial \xi_1 \partial \xi_2} - \frac{s_2}{C_2 - c_2}\frac{\partial w}{\partial \xi_1} - \frac{S_2}{C_2 - c_2}\frac{\partial w}{\partial \xi_2},$$

and

(10) $$4 \det (g_{\alpha\beta}) = \begin{vmatrix} d^2(C_2 - c_2) & 0 \\ 0 & d^2(C_2 - c_2) \end{vmatrix},$$

$g_{\alpha\beta}$ being the metric tensor of ξ_1, ξ_2; α, β ranging over 1, 2. The relations between the membrane stress components T^{11}, T^{22}, T^{12}, T^{21}, and the stress function F are

(11) $$\begin{aligned} T^{11} &= \frac{4}{d^4(C_2 - c_2)^2} F_{|22}, \\ -T^{12} = T^{21} &= \frac{4}{d^4(C_2 - c_2)^2} F_{|12}, \\ T^{22} &= \frac{4}{d^4(C_2 - c_2)^2} F_{|11}, \end{aligned}$$

where $T^{\alpha\beta}\, ds$ is the component, in the direction of increasing ξ_β, of the average membrane stress across the area formed by the normals to the middle surface of the plate along the element ds which is normal to a parametric line of ξ_α. The relations between the shearing stress, bending moment, and deflection w are

(12) $$\begin{aligned} -L^{11} = L^{22} &= \frac{4D}{d^4(C_2 - c_2)^2} w_{|12}, \\ L^{12} &= \frac{4D}{d^4(C_2 - c_2)^2} w_{|11}, \\ L^{21} &= \frac{-4D}{d^4(C_2 - c_2)^2} w_{|22}, \\ T^{10} &= \frac{4D}{d^4(C_2 - c_2)^2}\left(\frac{\partial}{\partial \xi_1}\Delta^* w - \frac{2S_2}{C_2 - c_2}\Delta^* w\right), \\ T^{20} &= \frac{4D}{d^4(C_2 - c_2)^2}\left(\frac{\partial}{\partial \xi_2}\Delta^* w - \frac{2s_2}{C_2 - c_2}\Delta^* w\right), \end{aligned}$$

where $L^{\alpha\beta}\, ds$ is the component, in the direction of increasing ξ_β, of the average bending moment across the area formed by the normals to the middle surface along the element ds which is normal to a parametric line of ξ_α; and $T^{\alpha 0}\, ds$ is the component, in the direction of increasing ξ_0, of the average bending moment across the area formed by the normals to the middle surface along the element ds which is normal to a parametric line of ξ_α. The restriction on the magnitudes of the above quantities for the Lagrange equations to be a good approximation is given by the following orders of magnitude with respect to the thickness $2h$:

$$\begin{aligned} O(w) &= O(h^n), & O(q) &= O(h^{n+3}), \\ O(F) &= O(h^{2n+1}), & O(T^{\alpha\beta}) &= O(h^{2n+1}), \\ O(T^{\alpha 0}) &= O(h^{n+3}), & O(L^{\alpha\beta}) &= O(h^{n+3}), \end{aligned} \tag{13}$$

where n is an integer greater than or equal to 2. We shall call the problems where the Lagrange equations are applicable small-deflection problems.

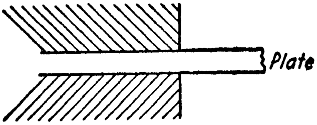

FIG. 1

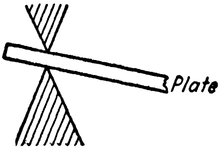

FIG. 2

The von Kármán plate equations for deflection of the order of the thickness of the plate expressed in elliptic coordinates are

$$\begin{aligned} D\Delta\Delta w &= -q + \frac{1}{\det(g_{\alpha\beta})}(w_{|11}F_{|22} + w_{|22}F_{|11} - 2w_{|12}F_{|12}), \\ \Delta\Delta w &= \frac{2h}{\det(g_{\alpha\beta})}(w_{|12}w_{|12} - w_{|11}w_{|22}). \end{aligned} \tag{14}$$

The expressions for the average membrane stress components, average shear stress components, and average bending moment components, in terms of w and F are the same for this case as for the Lagrange case. The restrictions on the magnitude of the above quantities for the von Kármán equations to be a good approximation are

$$\begin{gathered} O(w) = O(h), \qquad O(T^{\alpha 0}) = O(L^{\alpha\beta}) = O(h^4). \\ O(q) = O(F) = O(T^{\alpha\beta}) = O(h^3). \end{gathered}$$

We shall call the problems where the von Kármán equations are applicable large-deflection problems.

3. Edge conditions. When the plate is constrained at the edge so the deflection and its derivative normal to the boundary are zero, the plate is said to be clamped. When the plate is constrained so that the deflection and bending moment normal to the edge of the plate are zero at the boundary, the plate is said to be simply supported. Figures 1 and 2 represent a physical interpretation of the above conditions. In the figures, the shaded area indicates the support. If the plate is free to move laterally in the support, the plate is said to have free drift. Similarly, if the plate is constrained at the edge so that there is no lateral displacement, the edge condition is called no drift. For free drift, the components T^{11} and T^{12} of the average membrane stress are equal to zero at the edge of the plate. The boundary conditions that are easiest to formulate mathematically are: (A) the clamped plate with free drift and (B) the simply supported plate with free drift. The conditions on w and F for these two cases are

$$\tag{A} \begin{aligned} w(\bar{\xi}, \xi_2) &= 0, \qquad \left(\frac{\partial w}{\partial \xi_1}\right)_{\xi_1=\bar{\xi}} = 0, \\ (F_{|22})_{\xi_1=\bar{\xi}} &= 0, \qquad (F_{|12})_{\xi_1=\bar{\xi}} = 0, \end{aligned}$$

$$\tag{B} \begin{aligned} w(\bar{\xi}, \xi_2) &= 0, \qquad \left(\frac{\partial^2 w}{\partial \xi_1^2} - \frac{(1-\sigma)S_2}{C_2 - c_2}\frac{\partial w}{\partial \xi_1}\right)_{\xi_1=\bar{\xi}} = 0, \\ (F_{|22})_{\xi_1=\bar{\xi}} &= 0, \qquad (F_{|12})_{\xi_1=\bar{\xi}} = 0. \end{aligned}$$

When a general displacement is prescribed at the edge, the edge conditions contain both w and F and in addition are nonlinear.

4. Small deflections; Mathieu function solution. When the edge conditions are similar to the conditions (A) and (B) of Sec. 3, the small-deflection problem can be divided into two linear problems; *i.e.*, when two of the edge conditions depend on w alone, and are linear in w, and the two other edge conditions are linear in F but may also depend on w. The first Lagrange equation, $D\Delta\Delta w = -q$, with the two edge conditions in w alone, determine w. Once w is known, the second Lagrange equation, with the two edge conditions on F, becomes a problem of the same type. In this section we shall only be concerned with the divided type of problem mentioned above, namely,

$$\tag{15a} D\Delta\Delta w = -q,$$

(15*b*) Two linear boundary conditions in w.

We now look for a solution to the differential equation (15*a*) in the form $w = w_1 + w_2$, where w_1 is a particular solution of the above equation and w_2 is a biharmonic function chosen so that w assumes the appropriate values on the elliptical boundary of the plate. The following exhibits the particular solution, w_1, for a restricted class of functions q. Let us first consider the homogeneous problem:

$$\tag{16} \Delta f + k^2 f = 0.$$

In elliptic coordinates, this equation becomes

$$\tag{17} \Delta^* f + \frac{d^2 k^2}{2}(C_2 - c_2)f = 0,$$

where

$$C_2 = \cosh 2\xi_1, \qquad c_2 = \cos 2\xi_2,$$

$$\Delta^* f = \frac{\partial^2 f}{\partial \xi_1^2} + \frac{\partial^2 f}{\partial \xi_2^2}.$$

When $w(\xi_1, \xi_2) = u(\xi_1)v(\xi_2)$, it is found that equation (17) is satisfied if

$$\tag{18} u'' - (a - 2\lambda C_2)u = 0,$$

$$\tag{19} v'' + (a - 2\lambda c_2)v = 0,$$

where a is a constant and $\lambda = d^2k^2/4$. Equations (18) and (19) are called, respectively, the modified Mathieu equation and the ordinary Mathieu equation [5].

Let us now consider the Sturm-Liouville systems:

$$u'' - (a - 2\lambda C_2)u = 0, \tag{20a}$$

$$a_1u(\bar{\xi}) + a_2u'(\bar{\xi}) + a_3u(0) + a_4u'(0) = 0 \tag{20b}$$

$$b_1u(\bar{\xi}) + b_2u'(\bar{\xi}) + b_3u(0) + b_4u'(0) = 0, \tag{20c}$$

$$0 \leqq \xi_1 \leqq \bar{\xi}, \tag{20d}$$

$$v'' + (a - 2\lambda c_2)v = 0, \tag{21a}$$

$$v(\pi) - v(-\pi) = 0, \tag{21b}$$

$$v'(\pi) - v'(-\pi) = 0, \tag{21c}$$

$$-\pi \leqq \xi_2 \leqq \pi. \tag{21d}$$

Nontrivial solutions of the systems (20) and (21) are called modified Mathieu functions and ordinary Mathieu functions, respectively. It is found that the values of a, λ for which these nontrivial solutions exist form a denumerable set a_{mn}, λ_{mn}. The solutions to the simultaneous system (20), (21) are designated:

$$f_{mn} = \mathrm{Ce}_m\ (\xi_1, \lambda_{mn})\ \mathrm{ce}_m\ (\xi_2, \lambda_{mn}), \tag{22}$$

$$\bar{f}_{mn} = \mathrm{Se}_m\ (\xi_1, \bar{\lambda}_{mn})\ \mathrm{se}_m\ (\xi_2, \bar{\lambda}_{mn}). \tag{23}$$

These functions satisfy equation (16); thus,

$$\Delta f_{mn} = -4\frac{\lambda_{mn}}{d^2} f_{mn}$$

and

$$\Delta\Delta f_{mn} = \frac{16\lambda_{mn}^2}{d^4} f_{mn}.$$

When the loading function q can be expanded in a series of Mathieu functions of the form

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} P_{mn}\,\mathrm{Ce}_m\ (\xi_1, \lambda_{mn})\ \mathrm{ce}_m\ (\xi_2, \lambda_{mn}) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn}\,\mathrm{Se}_m\ (\xi_1\ \bar{\lambda}_{mn})\ \mathrm{se}_m\ (\xi_2, \bar{\lambda}_{mn}), \tag{24}$$

it follows that a particular solution to (15) is

$$w_1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\frac{d^4P_{mn}}{4^2D\lambda_{mn}^2}\,\mathrm{Ce}_m\ (\xi_1, \lambda_{mn})\ \mathrm{ce}_m\ (\xi_2, \lambda_{mn}) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} -\frac{d^4Q_{mn}}{4^2D\bar{\lambda}_{mn}^2}\,\mathrm{Se}_m\ (\xi_1, \bar{\lambda}_{mn})\ \mathrm{se}_m\ (\xi_2, \bar{\lambda}_{mn}). \tag{25}$$

Using the ordinary Sturm-Liouville theory, it can be shown that

$$\lambda_{mn} \geqq \frac{1}{8\,(1+\cosh 2\bar{\xi})}\left(m^2+\frac{n^2\pi^2}{\bar{\xi}^2}\right). \tag{26}$$

The orthogonality relation satisfied by the functions (22) and (23) and the expansion theorems are discussed by McLachlan [5] and others. The convergence of the solution (25) is indicated by the condition (26). The constants a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, are determined by the loading q. This resulting restriction on q is similar to the restriction for expanding a Fourier series. For example, for a constant load $a_1 = a_2 = a_3 = 0$, $a_4 = 1$, $b_1 = b_3 = b_4 = 0$, $b_2 = 1$.

When the interfocal distance $2d$ reduces to zero, the boundary ellipse reduces to a circle. At the same time, $\bar{\xi}$ the value of the parameter ξ_1 on the elliptical boundary approaches infinity. The parametric lines of ξ_1 reduce to the parametric lines r of polar coordinates. The parametric lines of ξ_2 reduce to the parametric lines of the polar coordinate θ. The particular solution (25) reduces to the Fourier-Bessel function series:

$$w_1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{P_{mn}}{k_{mn}^4}\,J_m\,(k_{mn}r)\cos 2m\theta + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{Q_{mn}}{k_{mn}^4}\,J_m\,(k_{mn}r)\sin 2m\theta. \tag{27}$$

A complementary solution w in the form

$$\begin{aligned} w_2 = \sum_{m=0}^{\infty} A_m\,(C_{m+2}c_m + C_m c_{m+2}) + B_m C_m c_m \\ + \sum_{m=1}^{\infty} D_m\,(S_{m+2}s_m + S_m s_{m+2}) + E_m S_m s_m, \end{aligned} \tag{28}$$

may be found using the method of Timpe [6, 7].

5. Small deflections; solution in terms of trigonometric and hyperbolic functions. Since very few numerical values of Mathieu functions have been tabulated, we shall now develop an equivalent solution in a series of hyperbolic and trigonometric functions. This solution is not so elegant as the former solution, but it lends itself more readily to numerical computation.

When the Laplacian operator is expressed in elliptic coordinates, equation (15a) becomes

$$\begin{aligned} C_2(\Delta^*\Delta^* w + 4\Delta^* w) + S_2\left(-4\,\frac{\partial\Delta^* w}{\partial\xi_1}\right) \\ + c_2(-\Delta^*\Delta^* w + 4\Delta^* w) + s_2\left(-4\,\frac{\partial\Delta^* w}{\partial\xi_2}\right) = -\frac{d^4(C_2-c_2)^3}{4D}\,q, \end{aligned} \tag{29}$$

where

$$\Delta^* = \frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}.$$

The form of the equation (29) suggests a series solution of the type

$$w = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}C_m c_n + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}S_m s_n, \tag{30}$$

where A_{mn}, B_{mn} are constants and

$$C_m = \cosh m\xi_1, \qquad S_m = \sinh m\xi_1, \qquad c_m = \cos m\xi_2, \qquad s_n = \sin m\xi_2.$$

By use of identities for hyperbolic and trigonometric functions of the type

$$C_2 C_m = C_{m+2} + C_{m-2}, \tag{31}$$

$$c_2 c_n = c_{n+2} + c_{n-2}, \tag{32}$$

the left-hand side of equation (29) with w replaced by the series (30) can be reduced to a series of the same type. Likewise, if q is also expressible as a series of this type, then the right-hand side of equation (29) can be changed into a similar series. The equations obtained by equating coefficients of similar terms on the two sides of the resulting equation then give recursion relations for the coefficients of the series (30). These recursion relations are linear and in general contain the four terms:

$$\begin{aligned} &A_{m-2,\,n}[(m-2)^2 - n^2][(m-4)^2 - n^2], \\ &+A_{m+2,\,n}[(m+2)^2 - n^2][(m+4)^2 - n^2], \\ &-A_{m,\,n-2}[(m)^2 - (n-2)^2][m^2 - (n-4)^2], \\ &-A_{m,\,n+2}[(m)^2 - (n+2)^2][m^2 - (n+4)^2]. \end{aligned}$$

It is found that they can be solved successively for A_m, B_m. The solution of these equations yields explicit values for A_{mn}, B_{mn} except when $m = n$, $|m - n| = 2$. This is to be expected since these indeterminate coefficients are the coefficients of terms which are biharmonic. These coefficients are then chosen so that the edge conditions are satisfied.

When the plate is circular, equation (30) reduces to the form

$$w = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A'_{mn} r^m \cos n\theta + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B'_{mn} r^m \sin n\theta. \tag{33}$$

6. Large deflections. The differential equations for the deflection w and stress function F, when the deflection is the order of the thickness $2h$, are equations (5) and (6). The extension of the method used in Secs. 4 and 5 to find the similar series solutions for these equations is direct. It is found that the equations that result from equating coefficients of corresponding terms, as in Secs. 4 and 5, are an infinite system of equations, each containing an infinite number of quadratic terms in the coefficients of the expansion for w and F. This system may be reduced to an infinite system of cubic equations for the coefficients of the expansion of w alone, each equation of which contains an infinite number of terms. These equations are similar to the ones obtained by Levy [8]. They may be solved approximately by the successive approximation method used by Levy. For elliptic plates, the coefficients of the terms in these equations are more cumbersome than those in Levy's equations. Thus it would entail considerable calculation to obtain approximate solutions by this method.

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NEW METHODS FOR THE ESTIMATION OF TORSIONAL RIGIDITY[1]

BY

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1. Introduction. This paper gives a review of some recent investigations on inequalities and estimates for torsional rigidity. Two methods will be presented which, while essentially different, have in common the property that they can be applied to other important problems in mathematical physics. The first method reduces the estimation of torsional rigidity to the estimation of Dirichlet's integral in Neumann's and Dirichlet's problem. The second method makes use of the idea of symmetrization and yields an isoperimetric inequality.

2. Notations and definitions. Let us consider a beam of infinite length and of uniform cross section. Let one such cross section be a domain B (which may be multiply connected) in the xy plane. If the beam is subjected to a twisting couple, each cross section is rotated through an angle which is proportional to its distance from the xy plane, and moreover the particles (x, y) of any cross section are displaced normally to its plane.

We begin by surveying the geometric data of our problem:

B is multiply connected plane domain.

m the order of connectivity of B. That is, the boundary of B consists of $m + 1$ curves.

$C_1, C_2, \cdots, C_m$, and C_0 are closed curves without double points which form the boundary of B. The domain B lies inside C_0 and outside C_i, for $i = 1, 2, \cdots, m$.

H_i is the interior of C_i (a hole).

A_i is the area of H_i.

$C = C_0 + C_1 + C_2 + \cdots + C_m$ is the full boundary of B.

$G = B + H_1 + H_2 + \cdots + H_m + C$ is a closed domain consisting of C_0 and its interior.

n is the *outer* normal to B at a variable point of C; the length of n is 1. The components of n are n_x and n_y.

The following functions of a variable point (x, y) in B are important in describing the physical situation:

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$\phi(x, y)$ the warping function,
$\psi(x, y)$ the function conjugate to $\phi(x, y)$,
$\Psi(x, y)$ the stress function.

The definitions of these functions will be discussed in the next sections.

3. The warping function and its conjugate. The function $\phi = \phi(x, y)$ represents, except for a constant factor, the vertical component of the displacement of the material point, the original position of which was at (x, y). We have put here and in the following the modulus of shear and angle of twist equal to 1. It is well known that the equilibrium of the elastic beam depends on the following conditions:

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{in } B, \tag{1}$$

$$\frac{\partial \phi}{\partial n} = \frac{\partial}{\partial s} \frac{x^2 + y^2}{2} \qquad \text{on } C. \tag{2}$$

We consider the analytic function $\phi + i\psi$ the real part of which is ϕ. If ϕ is given, ψ is determined except for an additive constant. Now ϕ is single valued, *i.e.*, uniquely defined at each point of the multiply connected domain B. It is not obvious that ψ is also single valued, but this fact is well established in the classical theory.[2]

Instead of expressing the conditions of equilibrium in terms of ϕ as we did in equations (1) and (2), we can express them also in terms of ψ. First,

$$\psi_{xx} + \psi_{yy} = 0 \qquad \text{in } B, \tag{3}$$

since ψ, conjugate to ϕ, is harmonic in D as is ϕ itself. Then, we obtain from (2)

$$\frac{\partial}{\partial s}\left[\psi - \frac{x^2 + y^2}{2}\right] = 0 \qquad \text{on } C. \tag{4}$$

Therefore, the function

$$\psi - \frac{x^2 + y^2}{2} = \Psi \tag{5}$$

is constant on each component part of C, *i.e.*, on each of the curves $C_1, C_2, \cdots, C_m$, and C_0, but it could take different constant values on different curves. In defining ψ from the warping function ϕ, we have an additive constant at our disposal, as we have observed above. We now dispose of this constant so that $\Psi = 0$ along C_0. Then the constant value of Ψ along C_i is uniquely determined; we call it k_i. Therefore,

$$\psi = \frac{x^2 + y^2}{2} \quad \text{on } C, \qquad \psi = k_i + \frac{x^2 + y^2}{2} \quad \text{on } C_i, \tag{6}$$

$$(i = 1, 2, \cdots, m).$$

---

[2] For the general theory here and in the following see, for instance, the textbooks on elasticity by A. E. H. Love and I. S. Sokolnikoff.

In order to solve the torsion problem for the cross section B, we can start either from equations (1) and (2) or from (3) and (4). In the former case, we seek ϕ and have a special case of Neumann's problem. In the latter case, we seek ψ and should have a special case of Dirichlet's problem if we would know the constants $k_1, k_2, \cdots, k_m$ from the start. Yet these constants are not given, and so the determination of ψ, although theoretically feasible, is less immediate. In the first part we shall use the function ϕ. Later we shall use the function Ψ and discuss its properties.

4. Definition of torsional rigidity or stiffness of a beam. If a beam of given cross section is twisted, the moment of the stresses about a given center per unit angle of twist is called the stiffness S of the beam. S is known to be independent of the choice of the center. The stresses in the beam, by Hooke's law, are expressed in terms of the strains which are, in turn, expressed in terms of the displacements, and therefore in terms of the warping function ϕ. Usually S is given by the formula

$$S = \iint [x^2 + y^2]\, dx\, dy + \iint \left[x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x}\right] dx\, dy, \tag{7}$$

where ϕ is the warping function, as yet unknown. (Another expression for S, due to Prandtl, will be used later here.) The first integral is the polar moment of inertia. For the second integral, we have by Green's formula, using the boundary conditions (2) for ϕ,

$$\begin{aligned} \iint \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x}\right) dx\, dy &= \iint \left[\frac{\partial (x\phi)}{\partial y} - \frac{\partial (y\phi)}{\partial x}\right] dx\, dy \\ &= \int_C \phi[xn_y - yn_x]\, ds = -\int_C \phi \frac{\partial \phi}{\partial n}\, ds = -\iint (\phi_x^2 + \phi_y^2)\, dx\, dy. \end{aligned} \tag{8}$$

We then have [1] the fundamental formula, replacing (7),

$$S = P - D(\phi), \tag{9}$$

where $D(\phi) = \iint (\phi_x^2 + \phi_y^2)\, dx\, dy$ is the Dirichlet integral.

It is interesting to note that this formula, whch results from a trivial and known transformation of a classical formula (7), has to our knowledge never been used for an investigation of the problem of torsional rigidity. It was only after the presentation of our results at the meeting of the American Mathematical Society in December, 1946, that a careful search kindly made by Professor T. J. Higgins among some 500 books and papers on this subject, revealed that the formula (9), or rather the formula (10), appears incidentally in one paper [4]. It is quite possible that the formula appears also somewhere else, but it is hardly probable that any significant use has been made of it.

To obtain S, we see that we need only to know $D(\phi)$, since the value of P

can be considered as known. Observing that $D(\phi) = D(\psi)$, where ψ is the conjugate of ϕ, we have also

$$S = P - D(\psi). \tag{10}$$

This formula could be useful only in the case of a simply connected domain. For the sake of uniformity, we shall consider mainly (9).

Since $D(\phi) \geqq 0$, we have for any (simply or multiply connected) section B,

$$S \leqq P. \tag{11}$$

The equality sign holds only when $D(\phi) = 0$, which implies that the warping function is a constant. As is well known, this can happen only when B is a circular region or a concentric circular ring.

The problem of finding upper and lower bounds for S has thus been reduced to the problem of finding lower and upper bounds for the Dirichlet integral in Neumann's and Dirichlet's problems, the last case corresponding to a simply connected domain.

We will consider mainly the more interesting case of a multiply connected domain, although the results will include the case of a simply connected cross section.[3] We will now develop a *general method* by which such bounds for D or, as a matter of fact, for any positive quadratic functional, can be obtained by a simple and uniform procedure based only on the Schwarz inequality and Green's formula [2]. The developments of the next paragraphs include a new elementary proof of several standard procedures, such as the Rayleigh-Ritz and the Trefftz methods, and applies to new cases which have been inaccessible up to now. Our procedure will also simplify some other recent methods for obtaining bounds for quadratic functionals of which Dirichlet's integral is the most outstanding example.

5. Schwarz's inequality and Green's formula. Putting for arbitrary functions ϕ and ψ,

$$D(\phi, \psi) = \iint_B (\phi_x \psi_x + \phi_y \psi_y)\, dx\, dy, \tag{12}$$

we have Schwarz's inequality

$$[D(\phi, \psi)]^2 \leqq D(\phi)D(\psi). \tag{13}$$

Besides this inequality, we shall employ the classical Green's formula, which is constantly used in the methods of Rayleigh-Ritz and Trefftz,

$$\iint_B \phi\Delta\psi\, dx\, dy + D(\phi, \psi) = \int_C \phi \frac{\partial\psi}{\partial n}\, ds, \tag{14}$$

---

[3] A lower but not an upper bound for S has been given by Courant [10]. Courant's method is essentially equivalent to our inequality (45).

$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ being the Laplacian, C the boundary of B, and n the outer normal to C.

6. Dirichlet's problem and Dirichlet's principle. Let us consider the Dirichlet problem: To determine $u(x, y)$ in B satisfying

$$\Delta u = 0 \qquad \text{in } B, \tag{15}$$

$$u = \text{a given function on } C. \tag{16}$$

The Rayleigh-Ritz method, which is essentially nothing else but Dirichlet's principle, furnishes an upper bound for $D(u)$. Let $w(x, y)$ take the same values as $u(x, y)$ on the boundary, *i.e.*,

$$w = u \qquad \text{on } C, \tag{17}$$

then we have, as known

$$D(u) \leqq D(w). \tag{18}$$

Let us show that this classical result, due to Riemann, can be obtained starting from Schwarz's inequality. We have

$$[D(u, w)]^2 \leqq D(u)D(w), \tag{19}$$

since by Green's theorem we have

$$D(u, w - u) = 0, \tag{20}$$

and

$$D(u, w) = D[u, u + (w - u)] = D(u). \tag{21}$$

Thus,

$$[D(u)]^2 \leqq D(u)D(w), \tag{22}$$

from which (18) follows, the trivial case $D(u) = 0$ being excluded.

7. Trefftz's method. Now we are going to show that the method of Trefftz [3], which furnishes a lower bound for $D(u)$, can also be easily deduced from Schwarz's inequality. Let $v(x, y)$ be a nonconstant function satisfying the equation

$$\Delta v = 0 \qquad \text{in } B; \tag{23}$$

then, according to Trefftz, we have

$$D(\lambda v) \leqq D(u), \tag{24}$$

λ being a suitably chosen real constant [see equation (28)]. In order to obtain (24), we proceed as in the preceding section. We have

$$[D(u, v)]^2 \leqq D(u)D(v), \tag{25}$$

and by Green's theorem

$$D(u, v) = -\iint_B u\Delta v \, dx \, dy + \int_C u \frac{\partial v}{\partial n} ds = \int_C u \frac{\partial v}{\partial n} ds. \tag{26}$$

Thus,

$$\frac{\left(\int_C u \frac{\partial v}{\partial n} ds\right)^2}{D(v)} \leqq D(u). \tag{27}$$

Since the left-hand side of (27) is known, we have here essentially the lower bound for $D(u)$ obtained by Trefftz. If we multiply and divide the left-hand side in (27) by $D(v)$ and set

$$\lambda = \frac{\int_C u \frac{\partial v}{\partial n} ds}{D(v)}, \tag{28}$$

we obtain (24) which is identical with the result formulated by Trefftz.

8. Neumann's problem. The technique outlined above may be applied to other boundary-value problems, and to some other cases provided the appropriate form of Schwarz's inequality is known. As an example, we consider Neumann's problem.

Let $u(x, y)$ be the solution of the boundary-value problem

$$\Delta u = 0 \qquad \text{in } B, \tag{29}$$

$$\frac{\partial u}{\partial n} = \text{a given function on } C, \tag{30}$$

the mean value of the given function being zero on C.

A lower bound for $D(u)$ may be obtained as follows: Let $v(x, y)$ be any nonconstant function. From Schwarz's inequality,

$$[D(u, v)]^2 \leqq D(u)D(v), \tag{31}$$

and from Green's formula,

$$D(u, v) = \int_C v \frac{\partial u}{\partial n} ds. \tag{32}$$

It follows that

$$\frac{\left(\int_C v \frac{\partial u}{\partial n} ds\right)^2}{D(v)} \leqq D(u). \tag{33}$$

Let us put

$$\frac{\int_C v \frac{\partial u}{\partial n} ds}{D(v)} = \mu,$$

where μ is a known quantity; then we have

$$D(\mu v) \leqq D(u), \tag{34}$$

which yields a lower bound for $D(u)$.

In order to obtain an upper bound for $D(u)$, we start with the following Schwarz's inequality:

$$\left[\iint_B (\phi_1 \psi_1 + \phi_2 \psi_2)\, dx\, dy\right]^2 \leqq \iint_B (\phi_1^2 + \phi_2^2)\, dx\, dy \iint_B (\psi_1^2 + \psi_2^2)\, dx\, dy. \tag{35}$$

Let $w_1(x, y)$ and $w_2(x, y)$ be two functions satisfying the conditions

$$\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0 \qquad \text{in } B, \tag{36}$$

and

$$w_1 n_x + w_2 n_y = \frac{\partial u}{\partial n} \qquad \text{on } C, \tag{37}$$

n_x and n_y being the components of the outer normal to C. We have, from (35),

$$\left[\iint_B (u_x w_1 + u_y w_2)\, dx\, dy\right]^2 \leqq D(u) \iint_B (w_1^2 + w_2^2)\, dx\, dy, \tag{38}$$

and by Green's theorem, using (36) and (37),

$$\iint_B (u_x w_1 + u_y w_2)\, dx\, dy = D(u). \tag{39}$$

Thus, the trivial case $D(u) = 0$ being excluded,

$$D(u) \leqq \iint_B (w_1^2 + w_2^2)\, dx\, dy, \tag{40}$$

which is Kelvin's theorem. In fact, $D(u)$ can be interpreted as the kinetic energy of an irrotational flow in B. The right-hand side of (40) is the kinetic energy in B of a rotational fluid having the same normal velocity on the boundary as the given irrotational flow. This theorem is valid in any number of dimensions.

For the two-dimensional case we can obtain, as is shown below, a new result seemingly not mentioned by Kelvin, by introducing the stream function $K(x, y)$ defined by

$$K_x = -w_2, \qquad K_y = w_1. \tag{41}$$

The integrability conditions are satisfied by (36). On C we have the boundary condition

$$K_y n_x - K_x n_y = \frac{\partial u}{\partial n}, \tag{42}$$

which can be written as

$$\frac{\partial K}{\partial s} = \frac{\partial u}{\partial n}. \tag{43}$$

In general, for a simply connected domain, the function K will be single valued, but in a multiply connected domain K will be a many-valued function. In all cases, however, the integral $D(K)$ is uniquely determined and is given by the formula

$$D(K) = \iint (K_y^2 + K_x^2)\,dx\,dy = \iint (w_1^2 + w_2^2)\,dx\,dy. \tag{44}$$

Therefore in the Neumann problem for two dimensions, we have, by (40),

$$D(u) \leqq D(K) \tag{45}$$

for a completely arbitrary $K(x, y)$ in B which satisfies the condition

$$\frac{\partial K}{\partial s} = \frac{\partial u}{\partial n} \quad \text{on } C.$$

The class of functions of type of K is not empty, since the conjugate to the harmonic function u can be taken as a function K. The inequality (45) we believe gives a new formulation of Kelvin's theorem.

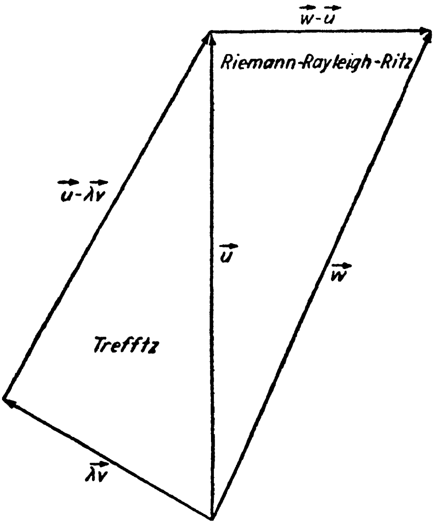

Fig. 1

9. Vector interpretation. Using the language of vector calculus, let us call a function a vector: Let us call $D(u, v)$ and $D(u)$ the scalar product of u and v and the square of the length of the vector u, respectively. We may then say that the Rayleigh-Ritz method consists in considering the unknown function (or vector) u as a side of a right triangle of which w is the hypotenuse, whereas in the Trefftz method u is taken as the hypotenuse of a right triangle having λv as a side (see Fig. 1).

In fact, it is easy to verify, using Green's formula, that $D(w - u, u)$ and $D(u - \lambda v, \lambda v)$ are both equal to zero, which expresses the orthogonality of the corresponding vectors as indicated in the figure. The analogy between functions and vectors is useful but, as is well known in other applications, to develop the concepts of function space, especially of a Hilbert space, more profound ideas are needed than those used in the preceding geometrical illustration.

10. Improvement of the bounds. Method of arbitrary parameters. This improvement is generally considered to be the essential part of the Rayleigh-Ritz method. Let us consider for instance, the upper bound for $D(u)$ in Dirichlet's problem. We had for any function w which is equal to u on C, the equation (20),

$$D(w) = D(u) + D(w - u) \tag{46}$$

which implies of course that $D(w) \geqq D(u)$.

The error of this upper bound is given by

$$D(w - u) = D(w) - D(u). \tag{47}$$

This error can be made smaller by the following procedure: Let us select $n + 1$ functions, $w^*, f_1, f_2, \cdots, f_n$ satisfying the following conditions:

$$w^* = u \qquad \text{on } C, \tag{48}$$

$$f_i = 0 \qquad \text{on } C \qquad (i = 1, 2, \cdots, n). \tag{49}$$

Let us put in (46) and (47)

$$w = w^* - \sum_{i=1}^{n} a_i f_i \tag{50}$$

with constant coefficients a_i. Obviously $w = u$ on C. By the general formula (47), the error will be now

$$\begin{aligned} D(w - u) = D\left[(w^* - u) - \sum_{i=1}^{n} a_i f_i\right] &= D(w^* - u) \\ &+ D\left(\sum_{i=1}^{n} a_i f_i\right) - 2D\left(w^* - u, \sum_{i=1}^{n} a_i f_i\right), \end{aligned} \tag{51}$$

or

$$D(w - u) = D(w^* - u) + \sum_{\substack{i=1,n \\ k=1,n}} a_i a_k D_{ik} - 2 \sum_{i=1}^{n} a_i D(w^* - u, f_i), \tag{52}$$

where $D_{ik} = D(f_i, f_k)$.

In these formulas w^* and f_i are fixed. The a_i's, however, are variable parameters.

By taking all the a_i's equal to zero, we have the error equal to $D(w^* - u)$. However, by taking the a_i's such that the right side of (52) is a minimum, we certainly obtain an error which is $\leqq D(w^* - u)$. By using formula (50), we have enlarged the class of admissible functions which are used for the computation of the upper bound $D(w)$ of $D(u)$ and have therefore decreased, or at least not increased, the minimum value of $D(w - u)$.

We will find the minimum of this function of the a's by a geometric consideration in a function space.

As we know, w^* can be interpreted as the hypotenuse in a right triangle where u is a leg. Since the functions f_i are zero on C, the function w in (50) can still be interpreted as a hypotenuse of another right triangle, in which u is again a leg.

Our problem is to select $a_1, \cdots, a_n$ in such a way as to make the hypotenuse w as short as possible (in the sense of the metric given by the Dirichlet integral). This can be done by taking the projection of w^* on the hyperplane determined by $f_1, f_2, \cdots, f_n$. (See Fig. 2 for $n = 2$.) We will show that this projection, which is uniquely determined, is given by

$$\sum_{i=1}^{n} a_i^0 f_i,$$

where the a_i^0's are the solutions of the equations

$$\sum_{k=1}^{n} a_k D_{ik} - D(w^* - u, f_i) = 0 \qquad (i = 1, \cdots, n), \tag{53}$$

and, as previously, we take for w the function

$$w = w^* - \sum_{i=1}^{n} a_i^0 f_i.$$

The projection of w^* on the hyperplane F, generated by the "vectors" $f_1, f_2, \cdots, f_n$, is a vector

$$\sum_{i=1}^{n} a_i^0 f_i$$

defined by the equations

$$D\left(w^* - \sum_{k=1}^{n} a_k^0 f_k, f_i\right) = 0 \qquad (i = 1, 2, \cdots, n), \tag{54}$$

which express the fact the vector

$$w = w^* - \sum_{k=1}^{n} a_k^0 f_k$$

is orthogonal to all vectors lying in F (see Fig. 2), because w is, by (54), orthogonal to each of the generating vectors f_i.

The system of the n equations (54) yields the following conditions for the determination of the parameters a_k^0:

$$\sum_{k=1}^{n} D_{ik} a_k^0 = D(w^*, f_i) \qquad (i = 1, 2, \cdots, n). \tag{55}$$

This system of linear equations is identical with the system (53) and has been obtained without the use of calculus. Instead of setting the partial derivatives with respect to the a_i's in (52) equal to zero, we have used here the fundamental property of the projection of w^* on F, namely, the fact that

$$D(w^* - \sum a_k^0 f_k) \leqq D(w^* - \sum a_k f_k) \tag{56}$$

for any values of the a_k's, the equality holding only for $a_k = a_k^0$, $k = 1, 2, \cdots, n$, which means geometrically that the square of the length of the vector $w^* - f^0$, perpendicular to the plane, is smaller than the length of any "oblique" vector (see Fig. 3).

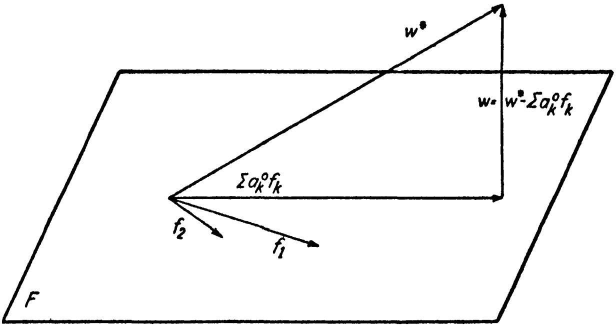

FIG. 2

In order to prove this, let us rewrite first (56) using obvious abbreviated notations, namely,

$$D(w^* - f^0) \leqq D(w^* - f). \tag{57}$$

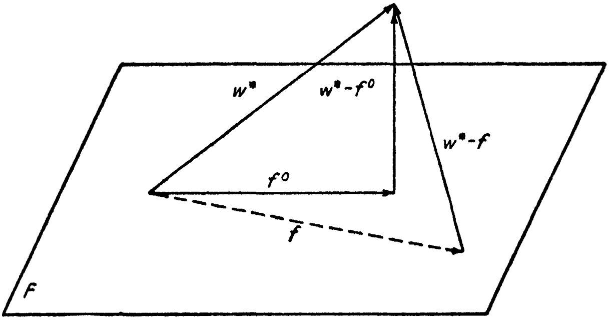

FIG. 3

We have

$$D(w^* - f) = D([w^* - f^0] + [f^0 - f]) \tag{58}$$
$$= D(w^* - f^0) + D(f^0 - f) + 2D(w^* - f^0, f^0 - f).$$

The last term is zero as by definition of f^0, the vector $w^* - f^0$ being orthogonal to F and therefore to $f^0 - f$. Therefore, we have from (58),

$$D(w^* - f) = D(w^* - f^0) + D(f^0 - f), \tag{59}$$

which proves (57) and shows that the equality sign holds for $f = f_0$ only.

We notice that in this proof, as in nearly every other method developed previously, we use Pythagoras' theorem for a function space, the square root of the Dirichlet integral being by definition the length of a vector.

The same remarks concerning the method of arbitrary parameters $a_1, a_2, \cdots, a_n$, apply also to the method of Trefftz as well as to the new method developed here for Neumann's problem.

11. The question of convergence. It is natural for the pure and for the applied mathematician to ask whether the bounds for D will converge if the number n of the given functions f_i is gradually increased. To investigate this question, let us first make, without loss of generality, a simplified assumption about the f_i's, namely, that they form an orthonormal system with the integral D used as a metric. We mean by this that we assume that the f_i's satisfy the following equations:

$$D(f_i, f_k) = 0 \qquad (i \neq k), \tag{60}$$

$$D(f_i, f_i) = 1. \tag{61}$$

In this case the a_i^0's are given by (55) as

$$a_i^0 = D(w^*, f_i) \qquad (i = 1, 2, \cdots, n). \tag{62}$$

The a_i^0's may be called the (generalized) Fourier coefficients of w^* with respect to the system f_i. As we have seen by (47) and (51), the error $D(w) - D(u)$ is equal to $D(w - u)$ or, since

$$w = w^* - \sum a_k^0 f_k,$$

the error is

$$D(w^* - u - \sum a_k^0 f_k). \tag{63}$$

Let us put

$$u_0 = w^* - u, \tag{64}$$

which implies that u_0 vanishes on C. By (63), we can write for the error the expression

$$D\left(u_0 - \sum_{k=1}^{n} a_k^0 f_k\right). \tag{65}$$

Let us note that

$$D(u_0, f_i) = D(w^* - u, f_i) = D(w^*, f_i) = a_i^0, \tag{66}$$

since $D(u, f_i) = 0$ by Green's formula,

$$D(u, f_i) + \iint f_i \Delta u \, dx\, dy = \int_C f_i \frac{\partial u}{\partial n} ds.$$

By (66) we see that the a_i^0's are also the Fourier coefficients of u_0 with respect to the f_i's. By (65), we see that the problem of convergence of the error to zero is reduced to the problem of approximation of an arbitrary function u_0 vanishing on the boundary by a linear combination of functions f_i which also vanish on the boundary. The error will tend to zero if we take for the f_i's a complete orthonormal sequence $f_1, f_2, \cdots$. The problem of convergence has thus been reduced to the problem of the construction of such sequences. Incidentally, we see that the question of convergence in all cases considered is equivalent to the question of the best approximation in the sense of the "mean square." The last term is used here in an extended way because the metric in all our problems has been given by $D(\phi)$ and not by $\iint \phi^2\,dx\,dy$.

12. Some critical remarks on the previous method. Trefftz's remarkable method has been widely used. Unfortunately, there are in modern literature some misleading statements about his results, which were developed for solving a particular problem, namely, to find a lower bound for $D(u)$ in Dirichlet's problem. An upper bound for $D(u)$ was found by the Rayleigh-Ritz method by taking a function w coinciding with u on C but satisfying no prescribed differential equation on the interior of B. A lower bound was found by Trefftz by taking a function h which is harmonic in the interior of B but which does not satisfy any prescribed boundary condition. The solution $u(x, y)$, which is known to be unique, is the uniquely determined function, the only element common to these two classes of functions, for it must have the properties of each class. Recently in connection with this particular problem solved by Trefftz, the statement has been repeatedly made that an upper bound for a functional can *always* be found by taking a function satisfying the prescribed boundary conditions of the corresponding problem and that a lower bound can be found by using a function which satisfies the prescribed differential equation. This is emphatically not the case as will be clearly seen, for instance, in the discussion of the Neumann problem given above. Trefftz is himself partly responsible for this latter misconception, as he calls his method a generalization of the "method of particular solutions," a rather cryptic statement.

Another widely diffused misunderstanding about the range of application of Trefftz's method is the opinion that it can be used to compute lower bounds of frequencies of vibrations. As is well known, these frequencies are given as the ratio of two quadratic functionals and not by one functional alone as in the previous case. This is a situation very different from the one just discussed, and in fact Trefftz himself never claimed that his method could be applied to this theory.

Trefftz's original proof was analyzed by Friedrichs [12] by the use of a Legendre transformation.

13. An application to torsional rigidity. As a simple application of our general theory, we derive an inequality for the torsional rigidity of the hollow square cross section bounded by the lines

$$x = \pm\alpha, \qquad y = \pm\alpha, \qquad x = \pm\beta, \qquad y = \pm\beta, \qquad (\beta > \alpha).$$

Here the boundary C consists of the outer square C_0 and the inner square C_1.[4]

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[4] For an approximate computation of S for a square cross section, see a paper by W. Prager and J. L. Synge [11].

According to the general theory (Sec. 8), we may take for K any function assuming on C_0 the same boundary values as $\frac{1}{2}(x^2 + y^2) - \frac{1}{2}\beta^2$ and on C_1 the same values as $\frac{1}{2}(x^2 + y^2) - \frac{1}{2}\alpha^2$. For instance, let us take (Fig. 4)

$$\begin{aligned} K(x, y) &= \tfrac{1}{2}y^2 \quad \text{in I and III,} \\ &= \tfrac{1}{2}x^2 \quad \text{in II and IV.} \end{aligned}$$

Clearly K has the prescribed boundary values, hence,

$$D(\phi) \leqq D(K). \tag{67}$$

For the domain under consideration,

$$P = \frac{8}{3}(\beta^4 - \alpha^4), \tag{68}$$

and a glance at the figure shows that

$$D(K) = 4 \iint_{\mathrm{I}} y^2\, dx\, dy = \frac{2}{3}(\beta^4 - \alpha^4). \tag{69}$$

Combining the last two equations with equation (9), we have finally

$$2(\beta^4 - \alpha^4) \leqq S \leqq \frac{8}{3}(\beta^4 - \alpha^4). \tag{70}$$

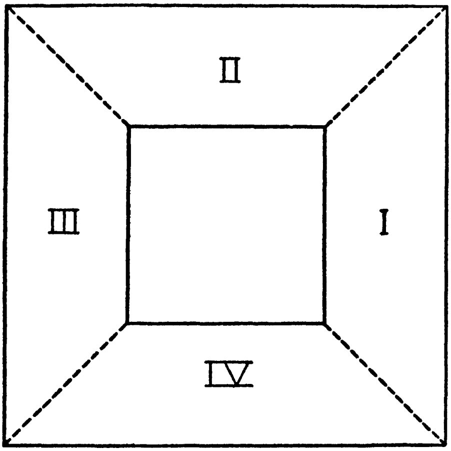

FIG. 4

14. An extremal property of torsional rigidity. We turn now to a second method for the estimation of torsional rigidity, which is *the method of symmetrization* which will lead us to the following result: *Of all multiply connected cross sections with given area and with given joint area of the holes, the ring bounded by two concentric circles has the maximum torsional rigidity.*

Let πa^2 be the joint area of the holes, $\pi(b^2 - a^2)$ the area of the cross section itself, and S the torsional rigidity or stiffness of the cylinder. Our theorem asserts that

$$S \leqq \frac{\pi}{2}(b^4 - a^4).$$

This theorem includes as a special case a celebrated proposition due to Saint-Venant which corresponds to the extreme case where there are no holes ($a = 0$). Saint-Venant's statement has been proved only recently by G. Polya by the method of symmetrization, used by G. Polya, G. Szego [5, 6, 7], and the present author.

In order to illustrate this result, we estimate the torsional rigidity S of the same frame as in Sec. 13. We have here the inequality

$$S \leqq \frac{8}{\pi}(\beta^4 - \alpha^4)$$

which is a slightly better upper bound than obtained in (70), but it should be noted that the numerical results of Sec. 13 can be improved, and also that lower bounds have been given.

Before introducing the method of symmetrization, we must make some further remarks on torsional rigidity.

15. Prandtl's stress function Ψ. We shall use here Prandtl's function Ψ, which we have defined by (5) and which is called the *stress function*. Its partial derivatives yield the components of stress

$$\Psi_y = \psi_y - y = \phi_x - y, \qquad -\Psi_x = -\psi_x + x = \phi_y + x. \tag{71}$$

Instead of basing the solution of the torsion problem on ϕ or on ψ, we could base it on Ψ which satisfies the following conditions:

$$\Psi_{xx} + \Psi_{yy} + 2 = 0 \qquad \text{in } B, \tag{72}$$

$$\Psi = 0 \qquad \text{on } C, \qquad \Psi = k_i \qquad \text{on } C_1, \qquad (i = 1, 2, \cdots, m). \tag{73}$$

Obviously, (72) is derived from (3) and (73) from (6). We need in the following some less obvious properties of the stress function Ψ.

I. It is well known that

$$\int_{C_i} \frac{\partial \Psi}{\partial n}\, ds = 2A_i. \tag{74}$$

A_i denotes, as defined in Sec. 2, the area of the hole H_i surrounded by C_i.

II. The function Ψ, continuous in $B + C$, attains its minimum at some point of this closed domain. At which point?

If the minimum of Ψ were attained at an inner point of B, we should have there

$$\Psi_x = \Psi_y = 0, \qquad \Psi_{xx} \geqq 0, \qquad \Psi_{yy} \geqq 0.$$

The last two conditions, however, contradict (72), and so the minimum of Ψ cannot be attained in the interior of B.

If the minimum of Ψ were attained at a point of C_i, it would be equal to k_i and attained all along C_i by (73). Therefore, we should have all along C_i

$$\frac{\partial \Psi}{\partial n} \leqq 0,$$

since n is the outer normal to B. This last inequality, however, contradicts (74), since $A_i > 0$, and so the minimum of Ψ cannot be attained on C_i.

As the other cases are excluded, the minimum of Ψ must be attained on C_0, and so it must be equal to 0, by (73). Thus, we have proved that

$$\Psi > 0 \qquad \text{in } B, \tag{75}$$

and that

$$k_1 > 0, \qquad k_2 > 0, \cdots, k_n > 0. \tag{76}$$

We shall see the foregoing results more intuitively if we extend the definition of Ψ to the whole domain G (see Sec. 2). We define

$$\Psi = k_i \qquad \text{in } H_i \qquad (i = 1, 2, \cdots, m), \tag{77}$$

and consider the surface the equation of which in rectangular coordinates x, y, and z is

$$z = \Psi(x, y). \tag{78}$$

We call the solid between this surface and the xy plane the "hill" (*Spannungshügel*). The slopes of the hill start from the contour line C_0 at level 0. The hill rises everywhere above its base, the domain G. The hill has m plateaus or tablelands of area $A_1, A_2, \cdots, A_m$, respectively. The hill may have a highest peak (a maximum above B) or a flat top (a maximum above one of the regions $H_1, H_2, \cdots, H_m$), but it has certainly no hollow in which rain water could gather (no minimum inside B).

III. The torsional rigidity or stiffness S is the value of the couple due to the action of the stresses distributed over the area of D. The stress components are given by (71). Therefore,

$$\begin{aligned} S &= \iint_B [x(-\Psi_x) - y\Psi_y]\,dx\,dy \\ &= -\iint_B \left[\frac{\partial(x\Psi)}{\partial x} + \frac{\partial(y\Psi)}{\partial y} - 2\Psi\right] dx\,dy \\ &= 2\iint_B \Psi\,dx\,dy - \sum_{\mu=1}^{m} k_\mu \int_{C_\mu} (xn_x + yn_y)\,ds \\ &= 2\iint_B \Psi\,dx\,dy + 2\sum_{\mu=1}^{m} k_\mu A_\mu = 2\iint_G \Psi\,dx\,dy. \end{aligned} \tag{79}$$

In passing from the double integral to the contour integral, we took into account (73), and used afterward (74) and (77). We see from (79) that $S/2$ is the volume of the hill.

16. Characterization of the torsional rigidity as a maximum. We consider now a function f that shares with Ψ some qualitative properties: f is defined and continuous throughout G, vanishes on the boundary C_0 of G, and takes the constant value c_μ in the region H_μ and on its boundary C_μ, for $\mu = 1, 2, \cdots, m$ (but c_μ and c_ν may differ if $\mu \neq \nu$). Finally, the partial derivatives f_x and f_y exist and are continuous, except possibly on a finite number of sufficiently smooth curves (as $C_1, C_2, \cdots, C_m$) and are bounded on the whole domain G.

We consider the integral

$$\begin{aligned} (f, \Psi) &= \iint_B (f_x\Psi_x + f_y\Psi_y)\,dx\,dy \\ &= \iint_B \left[\frac{\partial(f\Psi_x)}{\partial x} + \frac{\partial(f\Psi_y)}{\partial y} + 2f\right] dx\,dy \\ &= 2\iint_B f\,dx\,dy + \sum_{\mu=1}^{m} \int_{C_\mu} f\,\frac{\partial\Psi}{\partial n}\,ds \\ &= 2\iint_B f\,dx\,dy + 2\sum_{\mu=1}^{m} c_\mu A_\mu = 2\iint_G f\,dx\,dy, \end{aligned} \tag{80}$$

where we have used (72) and (74). Since Ψ is a particular function f, we obtain, as a particular case of (80),

$$(81) \qquad (\Psi, \Psi) = 2 \iint_G \Psi \, dx \, dy = S$$

by (79). By Schwarz's inequality, we obtain by the definition of (f, Ψ) that

$$(82) \qquad (f, \Psi)^2 \leqq (f, f)(\Psi, \Psi),$$

and so from (81) that

$$(83) \qquad \frac{\left(2 \iint_G f \, dx \, dy\right)^2}{\iint_G (f_x^2 + f_y^2) \, dx \, dy} \leqq S.$$

Equality is obviously attained when $f = c\Psi$ and c is a constant different from 0 [see again (81)]. A little discussion would show that there is no other case in which equality could be attained. At any rate, we have succeeded in characterizing S as the maximum of the left-hand side of (83). We should not forget, of course, that the functions f admissible in (83) are subject to the conditions stated at the beginning of the present section.

17. A particular case of a more general situation. In fact, the argument of the foregoing section is applicable to a more general situation. We could broaden the geometric conditions laid down at the beginning of Sec. 2 by admitting a disconnected domain D and multiply connected holes H_μ. The formulas of Sec. 16, correctly interpreted, would retain their validity under such wider assumptions. Yet we need not enter into such generalities. It will be enough to discuss the following simple particular case:

We are given $2n$ constants $\alpha_1, \alpha_2, \cdots, \alpha_n; \beta_1, \beta_2, \cdots, \beta_n$ which satisfy the condition

$$(84) \qquad 0 \leqq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n.$$

Let r denote, as usual, the distance of a variable point of the plane from the origin. We define G^*, C_0^*, H_ν^*, C_ν^*, and B^* by conditions dealing with r as follows:

$$\begin{aligned} &G^*: r \leqq \beta_n, \\ &C_0^*: r = \beta_n, \\ &H_\nu^*: \beta_{\nu-1} < r < \alpha_\nu \qquad && (\nu = 2, 3, \cdots, n), \\ &C_\nu^*: r = \beta_{\nu-1}, r = \alpha_\nu \qquad && (\nu = 2, 3, \cdots, n). \end{aligned}$$

If $\alpha_1 = 0$, H_1^* and C_1^* are not defined and are disregarded in the following discussion (or both are defined as the empty set). If $\alpha_1 > 0$, then

$$H_1^*: r < \alpha_1,$$

$$C_1^*: r = \alpha_1.$$

That is, G^* is a closed circular domain and C_0^* its boundary. For $\nu \geqq 2$, H_ν^* is an open circular ring and C_ν^* its boundary, consisting of two concentric circles. If H_1^* exists, it is an open circular domain and C_1^* is its boundary. Finally, we define

$$B^* = G^* - C_0^* - H_1^* - H_2^* - \cdots - H_n^* - C_1^* - C_2^* - \cdots - C_n^*.$$

That is, B^* is an open domain, consisting of disconnected parts unless $n = 1$. Only the case $n = 1$ falls under the assumptions of Sec. 2.

Under the present circumstances, the function Ψ, defined in G^* and satisfying (72) and (73) (read B^* for B and C_0^* for C_0), has the form

$$\Psi = \begin{cases} \text{const.} - \dfrac{r^2}{2} & \text{in } B^*, \\ \text{const.} & \text{in } H_\mu^*. \end{cases} \tag{85}$$

Different constants correspond to disconnected parts of B^* and also to H_μ^* and H_ν^* if $\mu \neq \nu$. These constants are so determined that Ψ is continuous through G^* although its analytic expression changes in crossing a boundary line of B^*, a circle with center O and radius α_ν or β_ν. We do not need here the easily obtainable explicit expressions of these constants in terms of $\alpha_1, \alpha_2, \cdots, \alpha_n$; $\beta_1, \beta_2, \cdots, \beta_n$. The hill representing Ψ is a solid of revolution with vertical axis passing through the origin. It has $n - 1$ ring-shaped plateaus and either a central peak or a flat top, according as $\alpha_1 = 0$ or $\alpha_1 > 0$. The double volume of the hill is

$$2\iint_G \Psi\, dx\, dy = \frac{\pi}{2}(\beta_1^4 - \alpha_1^4 + \beta_2^4 - \alpha_2^4 + \cdots + \beta_n^4 - \alpha_n^4) = S^*. \tag{86}$$

We can verify this by determining explicitly the constants intervening in (85), or we can derive (86) from the connection between the hill and the torsion problem, by observing the $\pi(\beta_\nu^4 - \alpha_\nu^4)/2$ is the torsional rigidity of a hollow cylinder with inner radius α_ν and outer radius β_ν.

The argument of Sec. 16 applies to the present geometric situation with obvious slight changes, and so we obtain the following result which corresponds to (83). If the function $f^* = f^*(x, y)$ vanishes along the circle C_0^* and is constant in each of the circular rings $H_1^*, H_2^*, \cdots, H_n^*$, but not constant throughout G^*, and smooth to the extent required in Sec. 16, then

$$\frac{\left(2\iint_{G^*} f^*\, dx\, dy\right)^2}{\iint_{G^*} (f_x^{*2} + f_y^{*2})\, dx\, dy} \leqq S^*. \tag{87}$$

We shall need later an estimate for S^*. We define a and b by

$$b = \beta_n, \tag{88}$$

$$a^2 = \alpha_1^2 - \beta_1^2 + \alpha_2^2 - \beta_2^2 + \alpha_3^2 - \cdots - \beta_{n-1}^2 + \alpha_n^2, \tag{89}$$

taking $a \geqq 0$. We shall prove that

$$S^* \leqq \frac{\pi}{2}(b^4 - a^4). \tag{90}$$

It may be worth while to emphasize the geometric meaning of a and b; πa^2 is the joint area of $H_1^*, H_2^*, \cdots, H_{n-1}^*$, and H_n^*, πb^2 the area of G^*, $\pi(b^2 - a^2)$ that of B^*. The inequality (90) can be made obvious by an appropriate diagram. In order to obtain a formal proof, we put

$$\beta_\nu^2 - \alpha_\nu^2 = p_\nu \qquad \beta_\nu^2 + \alpha_\nu^2 = q_\nu, \tag{91}$$

Observe that p_ν and q_ν are positive by (84).

From (88), (89), and (91) we obtain

$$\begin{aligned} b^2 - a^2 &= p_1 + p_2 + \cdots + p_n, \\ b^2 + a^2 &= q_n - p_1 - p_2 - \cdots - p_{n-1}. \end{aligned}$$

In view of the definition (86) of S^*, we can now transform the desired inequality (90) into

$$p_1q_1 + p_2q_2 + \cdots + p_nq_n \leqq (p_1 + \cdots + p_n)(q_n - p_1 - \cdots - p_{n-1}),$$

or finally into

$$(p_1 + p_2 + \cdots + p_n)^2 \leqq \sum_{\nu=1}^{n} p_\nu(q_n - q_\nu + p_n). \tag{92}$$

In order to prove this, we observe that

$$q_\nu - q_{\nu-1} = \beta_\nu^2 - \alpha_\nu^2 + \beta_{\nu-1}^2 - \alpha_{\nu-1}^2 + 2(\alpha_\nu^2 - \beta_{\nu-1}^2) > p_\nu + p_{\nu-1}, \tag{93}$$

by (90) and (91). It follows obviously from (93) that

$$\begin{aligned} p_n^2 &= p_n(q_n - q_n + p_n), \\ p_{n-1}(p_{n-1} + 2p_n) &< p_{n-1}(q_n - q_{n-1} + p_n), \\ p_{n-2}(p_{n-2} + 2p_{n-1} + 2p_n) &< p_{n-2}(q_n - q_{n-2} + p_n), \\ &\cdots\cdots\cdots\cdots \\ p_1(p_1 + 2p_2 + 2p_3 + \cdots + 2p_n) &< p_1(q_n - q_1 + p_n). \end{aligned}$$

Adding these inequalities, we obtain (92) which is equivalent to the desired (90). The proof shows also that the case of equality in (90) cannot arise if $n \geqq 2$.

18. Symmetrization with respect to a straight line. A (general) solid M and a solid of revolution M^* are in the following geometric relation: Any plane that is perpendicular to the axis of revolution of M^* and intersects one of the solids M and M^* intersects also the other, and the two intersections (which may be very different in form) are equal in area. If M and the axis of revolution are given, M^* is uniquely determined. We say that M is transformed into M^* by symmetrization with respect to the given axis. This kind of symmetrization was invented by H. A. Schwarz [8, 9] and we shall call it the "Schwarz symmetrization." Schwarz proved that the *surface area of M^* cannot be greater than that of M.* It is obvious, by Cavalieri's principle, that the volume of M^* is equal to that of M.

Applying the foregoing, we now symmetrize the "hill" considered in Sec. 15 with respect to the z axis. We obtain a solid of revolution which we call the "round hill," the equation of which we write as

$$(94) \qquad z = f^*(x, y).$$

The base of the hill, the region G, goes over into a circle, with equal area and with center at the origin, which is the base of the round hill and which we call G^*. In general, each plateau of the hill goes over into a plateau of the round hill, and these two plateaus have the same area. An exception may arise, however, if two or more different plateaus of the hill happen to have the same elevation; in such a case, their transforms by Schwarz symmetrization merge into a single annular tableland of the round hill. In any case, *the number of plateaus cannot increase, and their joint area remains unchanged by Schwarz symmetrization.*

We recognize here that the function arising in equation (94) of the round hill's surface fits exactly the description of the function f^* in Sec. 17. The number of annular tablelands of the round hill (n if $\alpha_1 > 0$, $n - 1$ if $\alpha_1 = 0$, according to Sec. 17) cannot be greater than the number of plateaus of the hill (m, according to Sec. 15) and so

$$(95) \qquad n \leqq m + 1.$$

The joint area of $H_1, H_2, \cdots, H_m$, *i.e.*, of the m plateaus of the hill, is exactly equal to the joint area of $H_1^*, H_2^*, \cdots, H_n^*$, and, therefore, equal to πa^2, according to the notation in Sec. 17. The areas of the bases, G and G^*, are also equal (equal to πb^2), and so are the volumes of the two hills, *i.e.*,

$$(96) \qquad \iint_G \Psi \, dx \, dy = \iint_{G^*} f^* \, dx \, dy.$$

By Schwarz's theorem, there is an inequality between the areas of the surfaces. As the areas of the bases are equal, it remains

$$(97) \qquad \iint_G [1 + \Psi_x^2 + \Psi_y^2]^{\frac{1}{2}} \, dx \, dy \geqq \iint_{G^*} [1 + f_x^{*2} + f_y^{*2}]^{\frac{1}{2}} \, dx \, dy.$$

We come now to a turning point in our argument. We introduce a (small) positive number ϵ and make the following remark: If the Schwarz symmetrization transforms the surface with equation (78) into that with equation (94), it also transforms the surface with equation $z = \epsilon\Psi$ into that with equation $z = \epsilon f^*$. Applying Schwarz's result to these latter surfaces, we obtain, instead of (97),

$$(98 \qquad \iint_G [1 + \epsilon^2(\Psi_x^2 + \Psi_y^2)]^{\frac{1}{2}} \, dx \, dy \geqq \iint_{G^*} [1 + \epsilon^2(f_x^{*2} + f_y^{*2})]^{\frac{1}{2}} \, dx \, dy.$$

Subtract from both sides of (98) the area of G, which is equal to that of G^*, divide the result by ϵ^2, and pass to the limit making $\epsilon \to 0$. Then we obtain

$$(99) \qquad \iint_G (\Psi_x^2 + \Psi_y^2) \, dx \, dy \geqq \iint_{G^*} (f_x^{*2} + f_y^{*2}) \, dx \, dy;$$

i.e., the Schwarz symmetrization diminishes Dirichlet's integral.

19. Proof of the main theorem. We start from the remark that (83) goes over into an equation if $f = \Psi$. This yields an expression for the torsional rigidity S of the multiply connected cross section B and the first link in the following chain of conclusions:

$$S = \frac{\left(2 \iint_G \Psi \, dx \, dy\right)^2}{\iint_G (\Psi_x^2 + \Psi_y^2) \, dx \, dy} \leqq \frac{\left(2 \iint_{G^*} f^* \, dx \, dy\right)^2}{\iint_{G^*} (f_x^{*2} + f_y^{*2}) \, dx \, dy}$$

$$\leqq S^* \leqq \frac{\pi}{2} (b^4 - a^4).$$

The second line is justified by (96) and (99), the third by (87), the last by (90), and so we obtain the result

$$S \leqq \frac{\pi}{2}(b^4 - a^4),$$

as announced in Sec. 14.

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STRESS FIELDS OF AXIALLY SYMMETRIC SHAFTS IN TORSION AND RELATED FIELDS

BY

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1. Introduction. In the following we consider solutions of the differential equations

$$\frac{\partial}{\partial r}\left(r^3\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial x}\left(r^3\frac{\partial\varphi}{\partial x}\right)=0, \tag{1}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial x}\left(\frac{1}{r^3}\frac{\partial\psi}{\partial x}\right)=0. \tag{2}$$

These equations are satisfied, respectively, by the angular displacement φ and the stress function ψ for axially symmetric shafts of variable radius in torsion. These states of stress are described more precisely by the displacement vector

$$\mathbf{v}=v\mathbf{j}=r\varphi\mathbf{j},\qquad \varphi=\varphi(x,r), \tag{3}$$

where r, θ, x are cylindrical coordinates, and **i**, **j**, **k** unit vectors in the direction of increasing r, θ, x. Since the displacement (3) is solenoidal,

$$\nabla\cdot\mathbf{v}=0, \tag{4}$$

its substitution into the elasticity equations with no body forces

$$\nabla^2\mathbf{v}+\frac{1}{1-2\sigma}\nabla(\nabla\cdot\mathbf{v})=0 \tag{5}$$

shows that $\mathbf{v}$ is harmonic:

$$\nabla^2\mathbf{v}=0. \tag{6}$$

Introducing cylindrical coordinates and noting that

$$\frac{\partial\mathbf{j}}{\partial r}=\frac{\partial\mathbf{j}}{\partial x}=0,\qquad \frac{\partial^2\mathbf{j}}{\partial\theta^2}=-\mathbf{j}, \tag{7}$$

there results for v the differential equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}=0, \tag{8}$$

and replacing v by $r\varphi$, one is led to equation (1).

Calculation of the various stress components associated with (3) shows that they vanish with exception of $p_{r\theta}$, $p_{x\theta}$, and these are given by

$$p_{r\theta}=rG\frac{\partial\varphi}{\partial r},\qquad p_{x\theta}=rG\frac{\partial\varphi}{\partial x}, \tag{9}$$

where G is the shear modulus. Expressing (1) in terms on $p_{r\theta}$, $p_{x\theta}$, one infers the existence of a "stress function" ψ such that

$$(10)\qquad p_{x\theta} = \frac{G}{r^2}\frac{\partial\psi}{\partial r}, \qquad p_{r\theta} = -\frac{G}{r^2}\frac{\partial\psi}{\partial x},$$

and comparing with (9) one obtains

$$(11)\qquad \frac{\partial\psi}{\partial r} = r^3\frac{\partial\varphi}{\partial x}, \qquad \frac{\partial\psi}{\partial x} = -r^3\frac{\partial\varphi}{\partial r}.$$

From equations (10) it may be shown that the cylindrical shell bounded by the surfaces $\psi = C_1$, $\psi = C_2$ is free from traction over these surfaces and transmits a torque $2\pi G\,\Delta\psi$ from one end to the other one.

Equation (1) is also of interest from the point of view of scalar solutions of the Laplace equation of the form

$$(12)\qquad V = V'(x, r)(\sin\theta, \cos\theta).$$

Indeed, $\nabla^2 V = 0$ leads to equation (8) for V', and hence to equation (1) for $\varphi = V'/r$. Again, as indicated above, equation (8) is satisfied by the magnitude v of any axially symmetric harmonic vector field of the form

$$(13)\qquad \mathbf{v} = v(x, r)\mathbf{j}.$$

This also follows by noting that the components of $\mathbf{v}$ are of the form (12):

$$(14)\qquad v_x = 0, \qquad v_y = -\,v(x, r)\cos\theta, \qquad v_z = v(x, r)\sin\theta.$$

Harmonic vector fields of the form (13) are of interest in connection with magnetic vector potentials of magnetic fields and induced electric fields due to axially symmetric electric currents.

Throughout the above (physical) three-dimensional Euclidean space is implied. However, equation (1) is also of interest from point of view of functions harmonic in *five*-dimensional Euclidean space. Indeed, it is satisfied by solutions of

$$(15)\qquad \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2} + \frac{\partial^2\varphi}{\partial x_4^2} = 0,$$

which are symmetric about the x axis, *i.e.*, such that

$$(16)\qquad \varphi = \varphi(x, r), \qquad r = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}.$$

In addition to the differential equations (1), (2), and (11) we also consider especially in Sec. 8, the differential equations

$$(17)\qquad \frac{\partial}{\partial r}\left(r^k\frac{\partial\varphi}{\partial r}\right) + \frac{\partial}{\partial x}\left(r^k\frac{\partial\varphi}{\partial x}\right) = 0,$$

$$(18)\qquad \frac{\partial}{\partial r}\left(r^{-k}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial x}\left(r^{-k}\frac{\partial\psi}{\partial x}\right) = 0,$$

$$\frac{\partial\psi}{\partial r} = r^k \frac{\partial\varphi}{\partial x}, \qquad \frac{\partial\psi}{\partial x} = -\, r^k \frac{\partial\varphi}{\partial r}, \tag{19}$$

where k is any integer, or even an arbitrary real number. For $k = 1$, equation (17) is identical with the three-dimensional Laplace equation for functions which are symmetric about an axis. For $k = 3$, equations (17) to (19), of course, reduce to equations (1), (2), and (11), respectively. For any integer k, equation (17) may be identified with the Laplace equation in $(k + 2)$-dimensional Euclidean space of coordinates x; $x_1, \cdots, x_{k+1}$, for functions which are symmetric about the x axis:

$$\varphi = \varphi(x, r), \qquad r = (x_1^2 + \cdots + x_{k+1}^2)^{\frac{1}{2}}. \tag{20}$$

Furthermore, equation (17) is also of interest from point of view of three-dimensional harmonic functions of the form

$$V = \cos m\theta \; V'(x, r). \tag{21}$$

By putting

$$V'(x, r) = r^m \varphi(x, r), \tag{22}$$

it is found that φ satisfies equation (17) for $k = 2m + 1$.

A strictly two-dimensional application of equations (1), (2), (11), and (17) to (19) is to conduction problems in thin plates where the conductivity or resistivity varies as the cube of the distance from the edge $r = 0$ or as r^k. If $-\varphi$ is identified with the electric potential, the conductivity is proportional to r^k, and ψ then represents the current flux function; if ψ is identified with the electric potential, then the resistivity varies as r^k.

2. Expansions in powers of r. Product solutions. One form of solution of equation (1) is obtained by expanding φ in powers of r^2,

$$\varphi = \varphi_0(x) + r^2\varphi_2(x) + r^4\varphi_4(x) + \cdots. \tag{23}$$

This assumes that φ is analytic, and an even function of r. Substituting in (1) and equating net coefficients of various powers of r to zero, one obtains

$$\begin{aligned} \varphi_0'' + 2.4\,\varphi_2 &= 0, \\ \varphi_2'' + 4.6\,\varphi_4 &= 0, \\ \cdots\cdots\cdots\cdots&\cdots, \end{aligned} \tag{24}$$

where primes denote differentiations with respect to x. Solving for $\varphi_2, \varphi_4, \cdots$, in terms of φ_0 and substituting in (23), there results

$$\varphi = \varphi_0 - \frac{r^2}{2\cdot 4}\varphi_0'' + \frac{r^4}{2\cdot 4\cdot 4\cdot 6}\varphi_0^{(4)} - \cdots. \tag{25}$$

Throughout the following it will be assumed that $\varphi_0(x)$ is analytic in x for real x.

A similar expansion may be obtained for ψ in powers of r. From (11) and (25) one obtains

$$\psi = \frac{r^4}{4}\varphi_0' - \frac{r^6}{2\cdot 4\cdot 6}\varphi_0^{(3)} + \frac{r^8}{2\cdot 4\cdot 4\cdot 6\cdot 8}\varphi_0^{(5)} \cdots; \tag{26}$$

where φ_0 is the *same function* of x as in equation (25).

If $\partial/\partial x$ is replaced by the symbol p, then equation (15) can be written in the form

$$(\nabla_4^2 + p^2)\varphi = 0, \tag{27}$$

where ∇_4^2 is the four-dimensional Laplacian,

$$\nabla_4^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}. \tag{28}$$

As shown in [1] equation (7), for a constant p^2, the solution of (27) which depends on r only, which is expansible in powers of r and is equal to 1 for $r = 0$, is given by

$$\varphi = 2J_1(rp)/(rp). \tag{29}$$

The expansion (25) may therefore be expressed symbolically in the form

$$\varphi(x, r) = 2[J_1(pr)/(pr)]\cdot\varphi_0(x), \qquad (p = d/dx). \tag{30}$$

This symbolic solution becomes a real solution if φ is of the product form

$$\varphi(x, r) = e^{\pm\alpha x}R(r), \tag{31}$$

$$\varphi(x, r) = (\cos\alpha x, \sin\alpha x)R(r), \tag{32}$$

whereupon p^2 may be replaced by α^2 or $-\alpha^2$ and there result the product solutions

$$\varphi = 2e^{\pm\alpha x}\, J_1(\alpha r)/(\alpha r), \tag{33}$$

$$\varphi = 2(\cos\alpha x, \sin\alpha x)\, I_1(\alpha r)/(\alpha r), \tag{34}$$

taking on along the x axis the values $\varphi_0 = e^{\pm\alpha x}$, $\varphi_0 = (\cos\alpha x, \sin\alpha x)$, respectively.

The solutions (26) corresponding to (33) and (34) (see [2], p. 327) are given by

$$\psi = \pm 2(r^2/\alpha)\, J_2(\alpha r)\, e^{\pm\alpha x}, \tag{35}$$

$$\psi = 2(r^2/\alpha)\, (\sin\alpha x, \cos\alpha x) I_2(\alpha r). \tag{36}$$

Applying harmonic analysis, one may obtain general solutions of equations (1) and (2) by summing solutions of the form (34) and (36) for various α.

3. Definite integral representation of solutions. An integral representation of general solutions of equations (1) and (2), involving an arbitrary analytic function of a complex variable, will now be deduced. Consider the function

$$\varphi_0(x + ix_1), \tag{37}$$

where φ_0 is an analytic function of the complex variable $x + ix_1$. This function satisfies the Laplace equation $\partial^2\varphi_0/\partial x^2 + \partial^2\varphi_0/\partial x_1^2 = 0$, and hence also the five-dimensional Laplace equations (15). From it a solution of (15) possessing the symmetry displayed in (16) and taking on the values $\varphi_0(x)$ along the x axis may be obtained by taking mean values of (37) over the loci (four-dimensional spheres) $\sum$:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 \qquad (x = \text{const.}), \tag{38}$$

(see [3], Theorem I, and its extension in Sec. 5). For details of integrations of this kind, the reader is referred to Sec. 6 of reference [1]. Applying the results obtained there for $m = 1$, $n = 4$, one obtains, upon replacing x_1 by s,

$$\varphi(x, r) = \frac{2}{\pi r^2}\int_{-r}^{r}\varphi_0(x + is)(r^2 - s^2)^{\frac{1}{2}}\, ds, \tag{39}$$

and introducing

$$\sin\theta = \frac{x_1}{r} = \frac{s}{r}, \tag{40}$$

$$\varphi(x, r) = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\varphi_0(x + ir\sin\theta)\cos^2\theta\, d\theta. \tag{41}$$

An alternative, purely algebraic, derivation of equation (39) is obtained by expanding φ_0 in a Taylor series in powers of s,

$$\varphi_0(x + is) = \varphi_0(x) + is\,\varphi'(x) - s^2\varphi''(x)/2 - \cdots, \tag{42}$$

substituting in (39), integrating term by term, and identifying the result termwise with (25). The direct method of establishing (39) and (41) has the advantage of being applicable even when (42) and (25) do not converge.

It is now readily shown from (11), (39), and (41) that similar integral representations for ψ are given by

$$\psi(x, r) = -\frac{2i}{\pi}\int_{-r}^{r}\varphi_0(x + is)s(r^2 - s^2)^{\frac{1}{2}}\, ds, \tag{43}$$

$$\psi(x, r) = -\frac{2i}{\pi}r^3\int_{-\pi/2}^{\pi/2}\varphi_0(x + ir\sin\theta)\sin\theta\cos^2\theta\, d\theta. \tag{44}$$

A further integral representation for solutions of (1), based on strictly three-dimensional considerations, may be obtained as follows: Starting with the harmonic function in the (x, y, z) space:

$$u = f(x + iy), \tag{45}$$

apply the linear functional operator (see [3], Secs. 4 and 5)

$$L_{1,2}(u) = \frac{\cos\theta}{\pi}\int_0^{2\pi} u\cos\theta'\, d\theta' \tag{46}$$

over the circles

$$x = \text{const.}, \qquad r = \text{const.} \tag{47}$$

to obtain harmonic functions of the form

$$\cos\theta \; V'(x, r). \tag{48}$$

The operator $L_{1,2}$ merely leads from u to the $\cos\theta$ term in the expansion of u in a Fourier series in θ. As explained in connection with (12), the function

$$\varphi(x, r) = \frac{V'(x, r)}{r} \tag{49}$$

is a solution of (1). Hence we obtain the following integral representation for φ:

$$\varphi(x, r) = \frac{1}{\pi r}\int_0^{2\pi} f(x + ir\cos\theta)\cos\theta\, d\theta, \tag{50}$$

where f is an arbitrary analytic function of its argument. It is readily shown that (50) is equivalent to (41), provided that

$$\varphi_0(x) = i\,f'(x). \tag{51}$$

From (11) and (50), one obtains

$$\psi(x, r) = -\frac{ir^3}{\pi}\int_0^{2\pi} f(x + ir\cos\theta)\cos^2\theta\, d\theta. \tag{52}$$

When the above integrals are applied to $\varphi_0 = e^{\alpha x}$, $\cos\alpha x$, $\cdots$, there result well-known integral representations of the Bessel functions in equations (33) to (36).

4. Spherical harmonics. Axially symmetric harmonic functions in three-space admit well-known expansions of the form

$$\sum A_n R^n P_n(\cos\omega), \qquad \sum B_n \frac{P_n(\cos\omega)}{R^{n+1}}, \tag{53}$$

convergent, respectively, inside and outside a proper sphere; here R is the distance from the origin, ω the angle of a radius vector with the axis of symmetry. Similar expansions exist in five-space for solutions of (15) possessing the symmetry (16); we again define R, ω as the distance from the origin, and the angle with the x axis:

$$R = (r^2 + x^2)^{\frac{1}{2}}, \tag{54}$$

$$\omega = \cos^{-1}\frac{x}{R} = \tan^{-1}\frac{r}{x}. \tag{55}$$

In three-space, the Legendre polynomials are often defined by the expansion, valid for $R < 1$, of R_1^{-1}, the inverse distance from $R = 1$, $\omega = 0$, in powers of R:

$$R_1^{-1} = (1 - 2\mu R + R^2)^{-\frac{1}{2}} = \sum_{n=0} P_n(\mu)R^n, \qquad \mu = \cos\omega. \tag{56}$$

The corresponding expansion for five-space is obtained by differentiating (56) with respect to μ, and dividing by R. There results

$$R_1^{-3} = (1 - 2\mu R + R^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} P'_n(\mu)R^{n-1} = \sum_{n=0}^{\infty} P'_{n+1}(\mu)R^n, \tag{57}$$

and this is valid for $R < 1$. The inverse cube of the distance from a point ($R = 1, \omega = 0$) is the fundamental harmonic function in five-space. Similarly, one obtains for $R > 1$,

$$R_1^{-3} = (1 - 2\mu R + R^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} \frac{P'_n(\mu)}{R^{n+2}} = \sum_{n=0}^{\infty} \frac{P'_{n+1}(\mu)}{R^{n+1}}, \tag{58}$$

either by expanding R_1^{-1} in negative powers of R, differentiating with respect to μ and dividing by R, or by applying five-dimensional spherical inversion to (57). In particular, for $\mu = 1$, one obtains from (57),

$$(1 - R)^{-3} = \sum P'_{n+1}(1)R^n, \tag{59}$$

whence

$$P'_{n+1}(1) = \frac{3\cdot 4\cdot \cdots (n + 2)}{n!} = \frac{(n + 2)(n + 1)}{2}. \tag{60}$$

From equations (57) and (58), it will be seen that in five-space the functions corresponding to the Legendre polynomials are $P'_{n+1}(\mu)$.

As in case of harmonic functions in three-space, it is readily shown that solutions of (15) are expansible in series of harmonic polynomials; hence solutions of (15), (16), or of (1) are expansible in series

$$\varphi = \sum_{n=0}^{\infty} A_n P'_{n+1}(\mu)R^n. \tag{61}$$

The expansion (61) is valid within the largest sphere, R = const., within which φ satisfies (1). Likewise, outside the smallest sphere beyond which φ satisfies (1), the following expansion holds:

$$\varphi = \sum_{n=0}^{\infty} B_n \frac{P'_{n+1}(\mu)}{R^{n+3}}, \tag{62}$$

provided that at infinity

$$\varphi = o(R^{-3}), \qquad \frac{\partial \varphi}{\partial R} = o(R^{-4}). \tag{63}$$

The coefficients A_n, B_n may be obtained by expanding $\varphi_0(x) = \varphi(x, 0)$ along the x axis in powers of x or $1/x$:

$$\varphi_0(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{64}$$

$$\varphi_0(x) = \sum b_n x^{-(n+3)}. \tag{65}$$

Recalling (60), one obtains

$$A_n = \frac{2a_n}{(n+2)(n+1)}, \qquad B_n = \frac{2b_n}{(n+2)(n+1)}. \tag{66}$$

The coefficients A_n, B_n may also be expressed in terms of boundary values of φ over a sphere $R =$ constant, utilizing the orthogonality of $P'_{n+1}(\mu)$ over such a sphere (in five-space), which may be given the form

$$\int_{-1}^{1} (1 - \mu^2)P'_{n+1}(\mu)P'_{m+1}(\mu)\, d\mu = 0 \qquad \text{for } n \neq m. \tag{67}$$

The "missing" harmonics varying as R^{-1}, R^{-2} will be obtained in Sec. 6.

Denoting the function given by the series (61) by $\varphi_5(R, \mu)$ and putting

$$\varphi_3(R, \mu) = \sum_{n=0}^{\infty} A_n P_{n+1}(\mu) R^{n+1}, \tag{68}$$

it will be seen that the following relations hold between φ_5 and φ_3:

$$\varphi_5 = \frac{1}{R}\frac{\partial \varphi_3}{\partial \mu} \qquad \text{for } \mu \neq \pm 1, \tag{69}$$

$$\varphi_5(R, 1) = \frac{d^2}{dR^2}\,[R\varphi_3(R, 1)]. \tag{70}$$

Now φ_3, except possibly for an additive constant missing in the series (68), is a general three-dimensional harmonic function symmetric about the x axis, while φ_5, similarly, is a general five-dimensional harmonic function which is symmetric about the x axis, *i.e.*, with the symmetry type displayed in equation (16). Equation (69) therefore shows that it is always possible to obtain a general five-dimensional harmonic function of this type of symmetry from a proper three-dimensional axially symmetric harmonic function. The above result is due to Timpe [5].

An alternative derivation of the solutions,

$$\varphi = P'_{n+1}(\mu)R^n, \tag{71}$$

$$\varphi = \frac{P'_{n+1}(\mu)}{R^{n+3}}, \tag{72}$$

and of the relation (69) is obtained by rewriting equations (11) in the form

$$\frac{\partial \psi}{\partial R} = R^2(1 - \mu^2)^2 \frac{\partial \varphi}{\partial \mu}, \qquad \frac{\partial \psi}{\partial \mu} = -R^4(1 - \mu^2)^2 \frac{\partial \varphi}{\partial R}. \tag{73}$$

This results, first by replacing the derivatives in directions of increasing x and r by $\partial/\partial R$, $\partial/R\partial\omega$, the derivatives in directions of increasing R, ω, then expressing $\partial/\partial\omega$ in terms of $\partial/\partial\mu$ by means of (55), and utilizing

$$\frac{r}{R} = \sin\omega = \sqrt{1 - \mu^2}. \tag{74}$$

Elimination of ψ from (73) leads to

$$\frac{\partial}{\partial \mu}\left[R^2(1-\mu^2)^2\frac{\partial \varphi}{\partial \mu}\right]+\frac{\partial}{\partial R}\left[R^4(1-\mu^2)\frac{\partial \varphi}{\partial R}\right]=0, \tag{75}$$

which is the equivalent of (1) in R, μ coordinates. Similarly, the equations

$$\frac{\partial \psi_3}{\partial R}=-(1-\mu^2)\frac{\partial \varphi_3}{\partial \mu}, \qquad \frac{\partial \psi_3}{\partial \mu}=R^2\frac{\partial \varphi_3}{\partial R}, \tag{76}$$

are obtained from (19) for $k = 1$ and apply to an axially symmetric three-dimensional harmonic function φ_3 and its Stokes (or flux) function ψ_3. Comparing equation (75) with the Laplace equation

$$\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \varphi_3}{\partial \mu}\right]+\frac{\partial}{\partial R}\left(R^2\frac{\partial \varphi_3}{\partial R}\right)=0, \tag{77}$$

which is obtained from (76), it is readily shown that equation (75) is satisfied by putting $\varphi = \varphi_5$ and using the relation (69) where φ_3 is a solution of equation (77).

The analogous relation to (69) for the function $\psi = \psi_5$ in (73) and ψ_3 in (76) is not so simple and is given by

$$\psi_5=-R\int\frac{(1-\mu^2)\,\partial^2\psi_3}{\partial \mu^2}\,d\mu+C, \tag{78}$$

where the integration is carried out for fixed R, and the constant of integration C is a proper function of R. In general, it is simpler to utilize (69) and equations (73) to obtain the φ_5 and the corresponding ψ_5. Thus one obtains

$$\psi=-(1-\mu^2)^2R^{n+3}\frac{P''_{n+1}(\mu)}{n+3}, \tag{79}$$

$$\psi=(1-\mu^2)^2\frac{P''_{n+1}(\mu)}{(n-1)R^{n-1}}, \tag{80}$$

corresponding to the solutions (71) and (72) of equation (1) (see [2, p. 326]).

Other product solutions of (1) and (11) are obtained from (71), (72), (79), and (80) by replacing P_n by Q_n, the Legendre functions of the second kind:

$$\varphi=Q'_{n+1}(\mu)R^n, \qquad \varphi=\frac{Q'_{n+1}(\mu)}{R^{n+3}}. \tag{81}$$

For $n = 1$, there results from (71) and (79), upon division by 3,

$$\varphi=x, \qquad \psi=\frac{r^4}{4}. \tag{82}$$

This corresponds to a long shaft of constant radius, transmitting a uniform torque.

Thus far the analysis of spherical harmonics was based on the analogy to five-dimensional harmonics. It can also be based on the connection described in Secs. 1 and 3 of solutions of (1) with three-dimensional harmonics [see equations (48) and (49)]. The expansions (57), (58), (59), etc., can now be arrived at by starting with (56) and differentiating with respect to y and z, utilizing relations of the form

$$\left(\frac{\partial}{\partial y} \pm i\frac{\partial}{\partial z}\right)[R^n P_n^m(\mu) e^{mi\theta}] = -R^{n-1}P_{n-1}^{m+1}(\mu)e^{i(m+1)\theta}, \tag{83}$$

for $m = 0$ (see [4], page 361, example 2).

5. Torque applied over boundary of shaft of uniform radius. As our first example, we consider the following boundary conditions applied over the boundary $r = a$ of a uniform shaft of radius a, of infinite length:

$$\psi\,|_{r=a} = H(x) = \begin{cases} 0 \text{ for } x < 0, \\ 1 \text{ for } x > 0, \end{cases} \tag{84}$$

where $H(x)$ is the "Heaviside unit function." This corresponds to the state of stress obtained in an infinitely long shaft of radius a by applying a torque over the infinitely thin ring $r = a$, $x = 0$ and an equal and opposite torque at $x = \infty$.

To obtain ψ by superposition of product solutions (35) and (36), we first write (84) as a Bromwich integral:

$$\psi\,|_{r=a} = \frac{1}{2\pi i}\int_L \frac{e^{px}}{p}\,dp, \tag{85}$$

where the integration is carried out in the p plane over a path L to the right of the imaginary axis. We next superpose solutions (35) thus:

$$\psi = r^2 \int A(p)e^{px}J_2(pr)\,dp, \tag{86}$$

choosing $A(p)$ so that (86) reduces to (85) for $r = a$. There results

$$\psi = \frac{1}{2\pi i}\frac{r^2}{a^2}\int_L \frac{e^{px}J_2(pr)}{pJ_2(pa)}\,dp. \tag{87}$$

The evaluation of (87) may be carried out by moving the path of integration to the left for $x > 0$, but to the right for $x < 0$. In the former case, the integral evaluates to $2\pi i$ times the sum of the residues over the poles of the integrand at $p = 0$ and at the roots of

$$J_2(pa) = 0 \tag{88}$$

on the negative real axis. There results

$$\psi = \frac{r^4}{a^4} + \frac{r^2}{a^2}\sum \frac{e^{-\xi_n x/a}J_2(\xi_n r/a)}{\xi_n J_2'(\xi_n)} \qquad (x > 0), \tag{89}$$

where the summation extends over the positive roots $\xi_1, \xi_2, \cdots$, of $J_2(\xi) = 0$.

For $x < 0$, there results similarly

$$\psi = -\frac{r^2}{a^2} \sum \frac{e^{\xi_n x/a} J_2(\xi_n r/a)}{\xi_n J_2'(\xi_n)} \qquad (x < 0), \tag{90}$$

where ξ_n are as in (89).

Denoting the right-hand members of (89) and (90) by $f(x, r)$, one can apply superposition as follows:

$$\psi = \int_{-\infty}^{+\infty} f(x - s, r)\, d\psi\, (s, a) \tag{91}$$

to obtain the solution ψ corresponding to any applied torque along $r = a$.

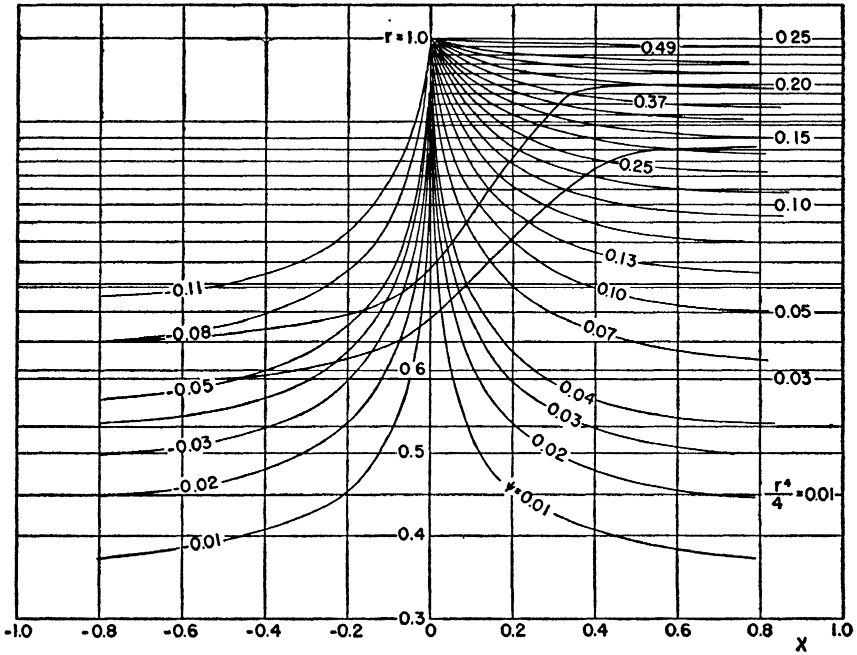

Fig. 1

A plot of the curves ψ = constant corresponding to equations (89) and (90) is given in Fig. 1. The convergence near $x = 0$ of the series (89) and (90) is quite poor, and the curves close to $x = 0$ were obtained by graphical interpolation from values obtained for larger numerical values of x. Furthermore, the constant $\frac{1}{2}$ was added to the function represented by (89) and (90) in order to obtain a resulting ψ which is odd in $x = 0$.

Figure 2 represents the curves ψ = constant for the function

$$\psi = \mp\frac{r^2}{a^3} \sum \frac{e^{-\xi_n|x|/a} J_2(\xi_n r/a)}{J_2'(\xi_n)} \qquad (x \lessgtr 0), \tag{92}$$

obtained by differentiation of the solution (87) to (90) with respect to x. The boundary values of this function ψ along $r = 1$ vanish with exception of $x = 0$,

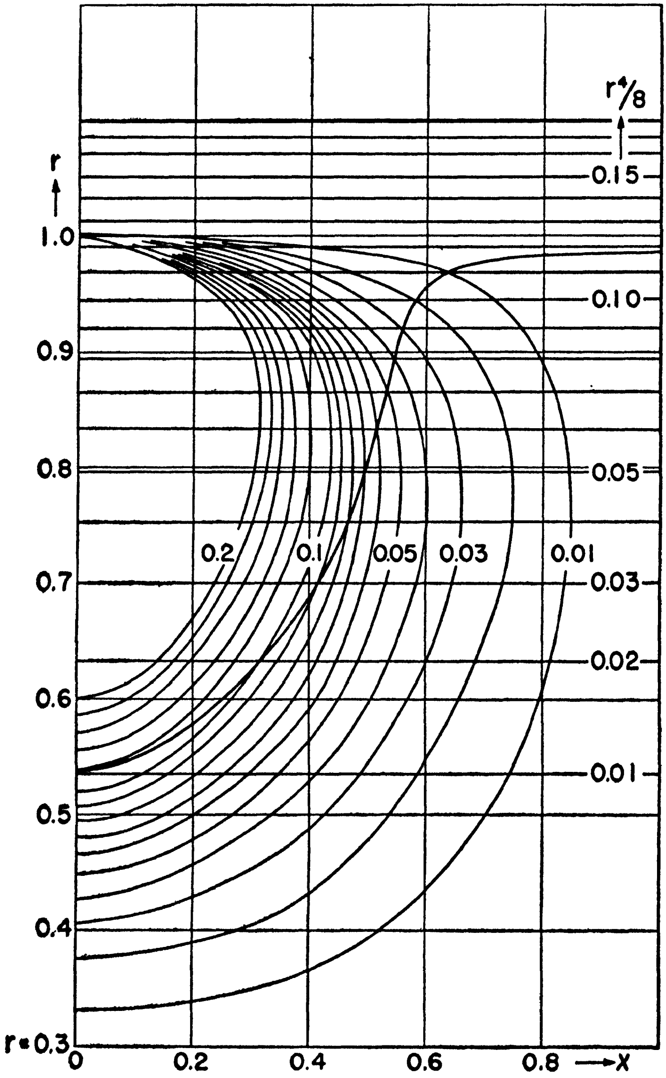

FIG. 2

near which point ψ becomes infinite, behaving at that point as the Dirac δ-function.

By superposition of the solutions (87) to (90) and of a uniform state of stress represented by equations (82), it is possible to obtain solutions corresponding to a shaft in torsion with a change of radius from one constant value at $x = -\infty$ to another constant value at $x = +\infty$. Two such curves ψ = constant are shown on Fig. 1. Similarly, by superposition of the solutions (92) and (82), boundaries are obtainable corresponding to a notch or a protuberance in a uniform shaft. Figure 2 shows a shape for a notch resulting from such a superposition.

6. Examples involving the integral representations. As an illustration of the integral representation method, we shall apply (44) to the special case

$$\varphi_0(x) = \frac{1}{x - a}, \tag{93}$$

where a is a constant, not necessarily real. There results

$$\psi = -\frac{i}{\pi} r^3 \int_0^{2\pi} \frac{\sin\theta \cos^2\theta \, d\theta}{x - a + ir\sin\theta}. \tag{94}$$

Introduction of

$$\lambda = e^{i\theta} \tag{95}$$

as a variable of integration converts the integral in (94) into

$$\int - \frac{(\lambda^2 - 1)(\lambda^2 + 1)^2 \, d\lambda}{4r\lambda^3[\lambda^2 - 1 + 2\lambda(x - a)/r]}, \tag{96}$$

where the integration is carried out in the positive sense (counterclockwise) over the unit circle $|\lambda| = 1$. The integral may be evaluated in terms of the residues at the two poles of the integrand inside $|\lambda| = 1$. At $\lambda = 0$, the residue is

$$-\frac{1}{4r}\left\{2 + \left[\frac{2(x - a)}{r}\right]^2\right\}. \tag{97}$$

At the other pole corresponding to the root of

$$\sin\theta = \frac{a - x}{ir} \tag{98}$$

lying in the upper half of the period strip in the θ plane, the residue may be obtained directly from (94) which yields for the residue

$$\frac{\sin\theta\cos^2\theta}{ir\cos\theta} = \frac{\sin\theta\cos\theta}{ir} = \frac{(x - a)}{r^2}\left[1 + \frac{(x - a)^2}{r^2}\right]^{\frac{1}{2}}. \tag{99}$$

Replacing the integral in (94) by the sum of (97) and (99) multiplied by $2\pi i$ yields

$$(100) \qquad \psi = -r^2 + 2(x-a)^2 + 2r(x-a)\left[1 + \frac{x - a^2}{r^2}\right]^{\frac{1}{2}}.$$

Similarly (41) and (93) yield

$$(101) \qquad \varphi = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos^2\theta\, d\theta}{x - a + ir\sin\theta} = -\frac{2(x-a)}{r^2} + \frac{2}{r}\left[1 + \frac{(x-a)^2}{r^2}\right]^{\frac{1}{2}}.$$

Further solutions may be obtained in a similar manner by starting with

$$(102) \qquad \varphi_0(x) = \frac{1}{(x-a)^2}, \quad \frac{2}{(x-a)^3}, \cdots .$$

They may also be obtained from (100) and (101) by differentiation with respect to a or x. Thus, corresponding to the first function in (102), one obtains

$$(103) \qquad \varphi = \frac{2}{r^2} + \frac{2(x-a)}{r^2\sqrt{r^2 + (x-a)^2}},$$

$$(104) \qquad \psi = 4(x-a) + 2\sqrt{r^2 + (x-a)^2} + \frac{2(x-a)^2}{\sqrt{r^2 + (x-a)^2}}.$$

In particular, for $a = 0$, one obtains from (101) and (103) solutions of (1) varying as R^{-1}, R^{-2}:

$$(105) \qquad \varphi = \frac{2x}{r^2} - \frac{2R}{r^2} = \frac{2}{R}\left[\frac{\mu}{(1-\mu^2)^{\frac{1}{2}}} - \frac{1}{1-\mu^2}\right],$$

$$(106) \qquad \varphi = \frac{2}{R^2}\left(\frac{1}{1-\mu^2} + \mu\right).$$

These supply the "missing" solutions in (72). The corresponding solutions of (2) are

$$(107) \qquad \psi = -r^2 + 2x^2 + 2xR = R^2(3\mu^2 + 2\mu - 1),$$

$$(108) \qquad \psi = 4x + 2R + \frac{2x^2}{R} = R(2\mu^2 + 4\mu + 2).$$

The stress fields represented by equations (102) to (105) are singular along the x axis. On the other hand, for nonreal a, the solutions (100), (101), (103), (104) and their x derivatives are analytic along the x axis.

By superposition of these solutions, one may obtain a variety of stress functions corresponding to rational $\varphi_0(x)$. In particular, the function ψ corresponding to

$$(109) \qquad \varphi_0(x) = x + \frac{A}{x^2 + b^2} + \frac{B}{(x^2 + b^2)^2}$$

represents a state of stress in a uniform shaft but with single "wave" (a radius decrease followed by an increase) near $x = 0$.

The term in (109) involving A may be obtained by replacing a in (93) by bi and noting that

$$\frac{1}{x - bi} = \frac{x + bi}{x^2 + b^2}, \tag{110}$$

whence $b/(x^2 + b^2)$ is seen to be the imaginary part of (110); a similar substitution of $a = bi$ in (100) and (101) and retention of the imaginary components yield

$$\varphi = -\frac{2b}{r^2} + \frac{2}{r} \operatorname{Im} \left\{ \left[1 + \frac{(x - ib)^2}{r^2} \right]^{\frac{1}{2}} \right\}, \tag{111}$$

$$\psi = +4xb + 2 \operatorname{Im} \{ (x - bi)[r^2 + (x - ib)^2]^{\frac{1}{2}} \} \tag{112}$$

for the corresponding functions φ, ψ.

Similarly, by utilizing

$$\frac{1}{(x^2 + b^2)^2} = \frac{1}{2b^2} \left[\frac{1}{x^2 + b^2} - \frac{(x^2 - b^2)}{(x^2 + b^2)^2} \right], \tag{113}$$

and resolving the last fraction through

$$\frac{x^2 - b^2}{(x^2 + b^2)^2} = \frac{1}{2} \left[\frac{1}{(x + bi)^2} + \frac{1}{(x - bi)^2} \right] = \operatorname{Re} \frac{1}{(x + bi)^2}, \tag{114}$$

one concludes that φ, ψ corresponding to (113) and (114) can be obtained from (111) and (112), and from (103) and (104) by replacing a by bi and taking real parts.

A plot of the curves ψ = constant for the equation (112) with $b = 1$ is given in Fig. 3. As noted, this corresponds to the function $\varphi_0(x) = 1/(x^2 + 1)$. The latter function was first considered as a possible shape by the superposition of which one could approximate to arbitrary functions $\varphi_0(x)$, thus enabling one to handle a variety of torsion problems. It turned out, however, that the function ψ possesses a singularity at $x = 0$, $r = 1$ and is discontinuous for $x = 0$, $r > 1$. For this reason, this function cannot be utilized for the purpose just mentioned, and a search was continued for other functions φ_0 of the same general shape as $1/(x^2 + 1)$ but leading to functions ψ without finite singularities.

Integrating (44) by parts, one obtains

$$\psi(x, r) = \frac{ir^3}{3\pi} \varphi_0(x + ir \sin \theta) \cos^3 \theta \Big]_{-\pi}^{\pi} + \frac{r^4}{3\pi} \int_{-\pi}^{\pi} \varphi_0'(x + ir \sin \theta) \cos^4 \theta \, d\theta. \tag{115}$$

The first term on the right-hand side generally will vanish. Further integration by parts yields

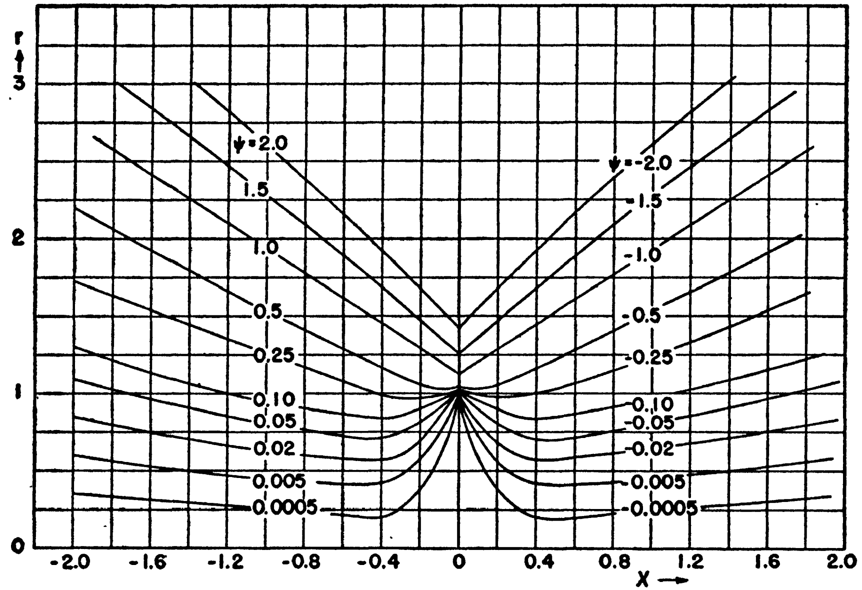

Fig. 3

$$\psi(x, r) = \frac{r^4}{4}\varphi_0'(x) - \frac{ir^5}{3\pi}\int_{-\pi}^{\pi}\varphi_0''(x + ir\sin\theta) \cdot \frac{\cos\theta}{32}[\sin 4\theta + 8\sin 2\theta + 12\theta]\,d\theta. \tag{116}$$

Equations (115) and (116) may be applied to multiple-valued functions such as

$$\varphi_0(x) = \ln x, \qquad \varphi_0(x) = x(\ln x - 1), \tag{117}$$

$$\varphi_0(x) = \iint \frac{(dx)^2}{(x^2 + b^2)^2}, \tag{118}$$

as well as to the function

$$\varphi_0(x) = \iint e^{-(x^2/a^2)}\,(dx)^2, \tag{119}$$

whose numerical values are not available for complex values. Superposition of (118), (119), and $\varphi = x$ corresponds to a shaft changing from one radius at $x = -\infty$ to another one at $x = +\infty$.

Applying (116) to the function

$$\varphi_0(x) = \frac{2}{\sqrt{\pi}}\int_0^x\int_0^x e^{-x^2}\,(dx)^2 \tag{120}$$

[this is a constant multiple of (119) with $a = 1$], one obtains

$$\varphi_0'(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx = \Phi(x), \tag{121}$$

and there results

$$\begin{aligned} \psi(x, r) &= \frac{r^4}{4}\, \Phi(x) \\ &\quad - \frac{2}{\sqrt{\pi}} \frac{ir^5}{96\pi} \int_{-\pi}^{\pi} e^{-(x+ir\sin\theta)^2} \cos\theta\, (12\theta + 8\sin 2\theta + \sin 4\theta)\, d\theta \\ &= \frac{r^4}{4}\, \Phi(x) - \frac{2}{\sqrt{\pi}} \frac{r^5 e^{-x^2}}{48\pi} \int_0^{\pi} e^{r^2 \sin^2\theta} \sin(2xr\sin\theta) \\ &\qquad \cdot \cos\theta\, (12\theta + 8\sin 2\theta + \sin 4\theta)\, d\theta. \end{aligned} \tag{122}$$

This integral was evaluated numerically, and the results were checked by means of the expansion (26) which yields

$$\begin{aligned} \psi &= \frac{r^4}{4}\, \Phi(x) - \frac{r^6}{2\cdot 4\cdot 6}\, \Phi_2(x) + \frac{r^8}{2\cdot 4\cdot 4\cdot 6\cdot 8}\, \Phi_4(x) \\ &\quad - \frac{r^{10}}{2\cdot 4\cdot 6\cdot 4\cdot 6\cdot 8\cdot 10}\, \Phi_6(x) + \cdots \end{aligned} \tag{123}$$

where Φ is defined in (121) and

$$\Phi_1(x) = \frac{d\Phi}{dx}, \qquad \Phi_2 = \frac{d\Phi_1}{dx}, \cdots.$$

The functions $\Phi(x), \Phi_1(x), \cdots, \Phi_6$ are tabulated in Jahnke-Emde, 1933 ed. pp. 24–29.

The curves $\psi =$ constant for equation (122) are plotted in Fig. 4. A superposition of this function and $\psi = r^4/4$ leads to possible shaft shapes with a change of radius shown on Fig. 5.

7. Solution of (1) with a logarithmic singularity. We shall now obtain a solution of (1) which is analytic everywhere except near (x_0, r_0) where it becomes infinite like

$$\tfrac{1}{2} \ln [(x - x_0)^2 + (r - r_0)^2]. \tag{124}$$

It will be sufficient to obtain φ for $x_0 = 0$, then replace $\varphi(x, r)$ by $\varphi(x - x_0, r)$. In terms of stresses, the solution corresponds to the application of a torque over the ring x_0, r_0, which, unlike the ring in the example of Sec. 5, is internal to the stressed body. The manner in which the applied torque is held in equilibrium at infinity will be explained presently.

To obtain φ, we shall utilize the relation of φ to axially symmetric harmonic vector fields **v** of the form (3) described in Sec. 1. We start with the following integral for **v** (see Fig. 6):

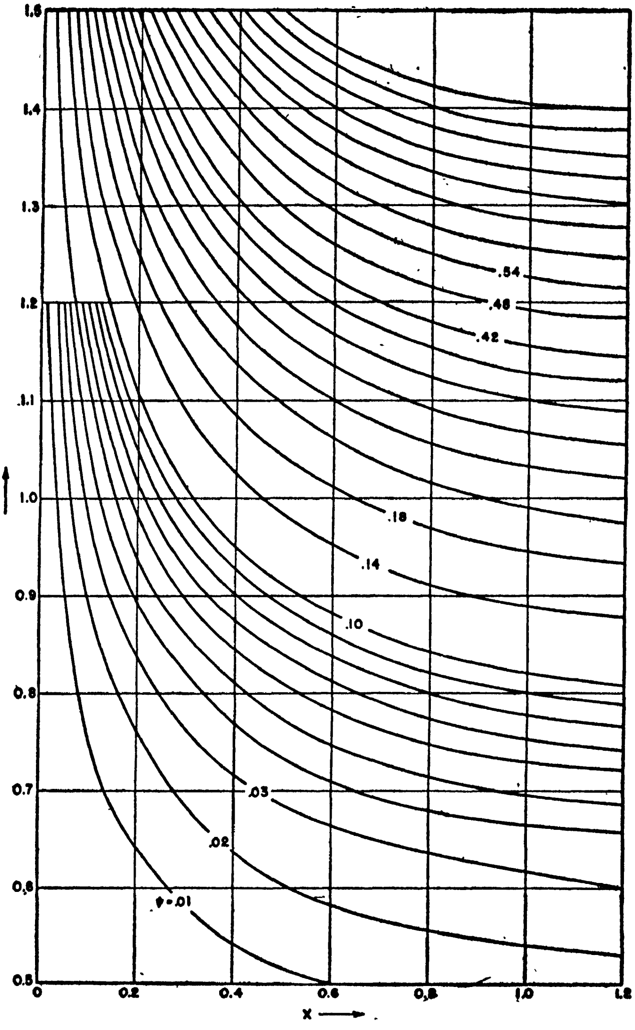

FIG. 4

$$\mathbf{v} = \int \frac{\mathbf{u}}{R'}\, ds, \tag{125}$$

where the integration is carried out over the circle

$$r = r_0, \qquad x = 0, \tag{126}$$

ds being the element of length along the path of integration, $\mathbf{u}$ the unit tangent vector along it, and R' the distance from the field point P to the point Q on the integration path. Since $1/R'$ is harmonic in the coordinates of P, the integral in (125) is harmonic; from symmetry, the integral $\mathbf{v}$ is seen to be of the form (3). The vector field $\mathbf{v}$ corresponds to the magnetic vector potential of a circular unit current flowing over the ring of integration. From familiar potential theory, it follows that v becomes infinite near the ring like

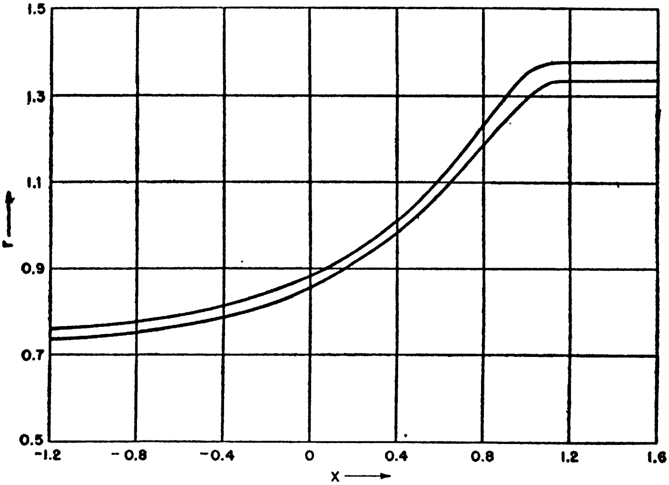

FIG. 5

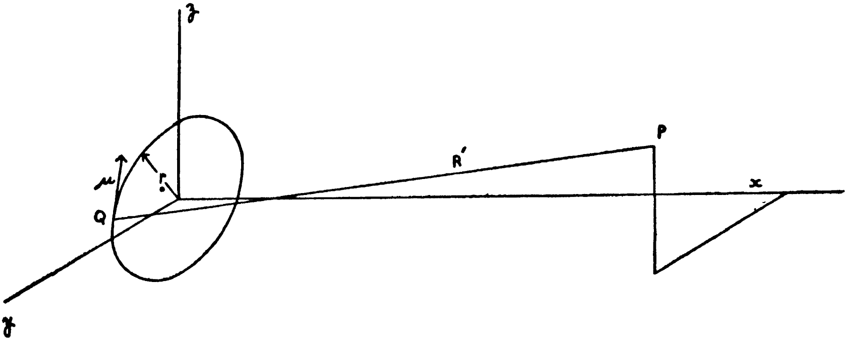

FIG. 6

$$(127) \qquad 2 \ln \rho,$$

where ρ is the (shortest) distance from the ring. To evaluate v, we place P in the half plane $z = 0$, $y > 0$ (or $\theta = 0$), whereupon $\mathbf{v}$ has the direction of the z axis, and there results upon replacing $\mathbf{u}$ by

$$(128) \qquad \mathbf{u} = -\mathbf{i}_y \sin \theta' + \mathbf{i}_z \cos \theta',$$

where $\mathbf{i}_y$, $\mathbf{i}_z$ are unit vectors along the y and z axes,

$$\mathbf{v} = \mathbf{i}_z r_0 \int_0^{2\pi} \frac{\cos\theta'\, d\theta'}{[x^2 + y^2 + r_0^2 - 2yr_0 \cos\theta']^{1/2}} \tag{129}$$

and hence, for any position of P,

$$v(x, r) = r_0 \int_0^{2\pi} \frac{\cos\theta\, d\theta}{[x^2 + r^2 + r_0^2 - 2rr_0 \cos\theta]^{1/2}}. \tag{130}$$

The integral is elliptic and can be reduced to the following form:

$$v = 2\sqrt{\frac{r_0}{r}}\left[\left(\frac{2}{K} - k\right)K - \frac{2}{k}E\right], \tag{131}$$

where K, E are the complete elliptic integrals of the first and second kind, and

$$k^2 = \frac{4rr_0}{(r + r_0)^2 + x^2}. \tag{132}$$

Recalling (3), we obtain upon multiplication by $r_0/2r$

$$\varphi = \frac{vr_0}{2r} = \left(\frac{r_0}{r}\right)^{3/2}\left[\left(\frac{2}{k} - k\right)K - \frac{2}{k}E\right]. \tag{133}$$

For $r = 0$, v vanishes by symmetry, but φ does not, and the expansion of the integrand in (130) in powers of r yields for $\varphi_0(x) = \varphi(x, 0)$.

$$\varphi_0(x) = \frac{\pi r_0^{3/2}}{(x^2 + r_0^2)^{3/2}}; \tag{134}$$

this is also obtained from (132) and (133) by expanding in powers of k^2 and r.

A. Weinstein [7] has obtained expressions for the above function φ and the corresponding function ψ in terms of Bessel integrals. The finite form (133) is more convenient from a computational point of view.

Applying equations (41) and (44) to the function φ_0 given by (134), one obtains

$$\varphi(x, r) = r_0^3 \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta\, d\theta}{[(x + ir\sin\theta)^2 + r_0^2]^{3/2}}. \tag{135}$$

$$\psi(x, r) = -ir^3 r_0^3 \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta \sin\theta\, d\theta}{[(x + ir\sin\theta)^2 + r_0^2]^{3/2}}. \tag{136}$$

These are elliptic integrals and can be reduced to standard forms by replacing $\sin\theta$ by a new variable μ, then using a linear fractional transformation on the latter:

$$\frac{u - 1}{u + 1} = e^{i\alpha}\frac{\mu - 1}{\mu + 1}, \tag{137}$$

where

$$\alpha = \arg\frac{r + r_0 - ix}{r - r_0 + ix}.$$

The parameter k_1 that results is given not by equation (132) but by

$$k_1 = \frac{[(r_0 + r)^2 + x^2]^{\frac{1}{2}} - [(r_0 - r)^2 + x^2]^{\frac{1}{2}}}{[(r_0 + r)^2 + x^2]^{\frac{1}{2}} + [(r_0 + r)^2 + x^2]^{\frac{1}{2}}}. \tag{138}$$

It is related to k in (132) by the Landen transformation:

$$k_1 = \frac{1 - k'}{1 + k'}, \qquad k' = (1 - k^2)^{\frac{1}{2}}. \tag{139}$$

The function ψ is multiple-valued near the point $(0, r_0)$ where φ has a logarithmic singularity, behaving there like

$$\tan^{-1} \frac{x}{r - r_0} \tag{140}$$

and increasing by 2π for a path encircling the point $(0, r_0)$.

Expansions for φ and (proper determinations of) ψ in spherical harmonics are obtained by starting with

$$\varphi_0(x) = \frac{\pi}{2}\left[1 - \frac{3}{2}\frac{x^2}{r_0^2} + \frac{3}{2}\frac{5}{2}\frac{1}{2!}\frac{x^4}{r_0^4} - \cdots\right] \qquad (|x| < r_0), \tag{141}$$

$$\varphi_0(x) = \frac{\pi}{2}\left[\left(\frac{r_0}{x}\right)^3 - \frac{3}{2}\left(\frac{r_0}{x}\right)^5 + \frac{3}{2}\frac{5}{2}\frac{1}{2!}\left(\frac{r_0}{x}\right)^7 - \cdots\right] \qquad (|x| > r_0), \tag{142}$$

and utilizing equations (61) to (66). There results

$$\varphi(x, r) = \pi\left[\frac{1}{2} - \frac{3}{2}\frac{P_3'(\mu)}{3\cdot 4}\frac{R^2}{r_0^2} + \frac{3}{2}\frac{5}{2}\frac{1}{2!}\frac{P_5'(\mu)}{5\cdot 6}\frac{R^4}{r_0^4} - \cdots\right] \qquad (R < r_0), \tag{143}$$

$$\varphi(x, r) = \pi\left[\frac{1}{2}\left(\frac{r_0}{R}\right)^3 - \frac{3}{2}\frac{P_3'(\mu)}{3\cdot 4}\left(\frac{r_0}{R}\right)^5 + \cdots\right] \qquad (R > r_0). \tag{144}$$

In particular, the first term of equation (144), along with equations (9) or an equivalent set of equations derived from (73), yields the stress distribution for large R and exhibits the manner in which the applied torque over the circle $x = 0$, $r = r_0$ is held in equilibrium. Likewise, by replacing the terms (71) and (72) in the above expansions by (79) and (80), one obtains the corresponding expansion for $\psi(x, r)$ in powers of R.

As a further check on the preceding equations, if one replaces the polynomials $P_3'(\mu)$, $P_5'(\mu) \cdots$ in (143) by their explicit expansions, then introduces x^2 for $\mu^2 R^2$, each term of (143) becomes a homogeneous polynomial in x^2, r^2; by rearranging terms in powers of r^2, the result may be checked against the expansion (25) for φ_0 given by (134) [these expansions agree with similar expansions of the integrand in (135) in powers of r], and against corresponding expansions in powers of r derived from the integral (130).

Application of the equation

$$\iint [\varphi L(\bar{\varphi}) - \bar{\varphi} L(\varphi)]\, dx\, dr \tag{145}$$

$$= \int \varphi\left(r^3 \frac{\partial \bar{\varphi}}{\partial r}\, dx - r^3 \frac{\partial \bar{\varphi}}{\partial x}\, dr\right) - \bar{\varphi}\left(r^3 \frac{\partial \varphi}{\partial r}\, dx - r^3 \frac{\partial \varphi}{\partial x}\, dr\right)$$

enables one to express φ at (x_0, r_0) in terms of boundary values of φ and $\partial\varphi/\partial n$. Here L is the differential operator in equation (1), φ is any solution of (1), $\bar{\varphi}$ is the solution of (1) given by (133) but with x replaced by $x - x_0$, and the integrations are carried out over any area in the x, r plane and over its boundary.

8. Extensions to certain other differential equations. We turn now to the solutions of the $(k + 2)$-dimensional equations (17) to (19), and to the extension of some of the results obtained above for the equations (1), (2), and (11) to these equations.

A solution of (17) in powers of r^2, analogous to (25), is given by

$$\varphi = \varphi_0(x) - \frac{r^2 \varphi_0''(x)}{2(k+1)} + \frac{r^4 \varphi_0^{(4)}(x)}{2 \cdot 4(k+1)(k+3)} - \cdots. \tag{146}$$

An integral representation of solutions of equation (17) taking on the values $\varphi_0(x)$ on the x axis and involving the analytic function $\varphi_0(x + ir)$ can be obtained in a manner similar to (39) and (41) by taking means of $\varphi_0(x + ix_1)$ over the sphere $\sum$:

$$x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = r^2, \qquad x = \text{const.}, \tag{147}$$

and is given by

$$\begin{aligned} \varphi(x, r) &= \int_{-r}^{r} \varphi_0(x + is)(r^2 - s^2)^{(k-1)/2} \frac{ds}{r^k C_{(k-1)/2}} \\ &= \int_{-\pi/2}^{\pi/2} \varphi_0(x + ir \sin\theta) \cos^{(k+1)/2}\theta \frac{d\theta}{C_{(k-1)/2}}, \end{aligned} \tag{148}$$

where

$$C_n = \int_{-\pi/2}^{\pi/2} \cos^n\theta \, d\theta = \frac{\pi}{2^n} \frac{n!}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!},$$

$n!$ representing the factorial of n, $n! = \Gamma(n + 1)$.

An integral representation for φ similar to (50) may be obtained by utilizing relations (21) and (22) and applying the operator $L_{1,m}$ (see previous reference to [3]) over the circles r = constant, x = constant. There results

$$\varphi(x, r) = \frac{1}{\pi r^m} \int_0^{2\pi} f(x + ir\cos\theta) \cos m\theta \, d\theta. \tag{149}$$

Similar power series and integral representation may be obtained for solutions ψ of (18) and (19).

Of special interest is, of course, the case $k = 1$ when (17) becomes the Laplace equation for three-dimensional and axial symmetry, while the function ψ becomes the Stokes or flux function for fields whose potential is φ.

Returning to general values of k, spherical harmonic functions and expansions similar to (57) amd (58) may be readily obtained by further μ differentiation of the expansion (56) and division by R for odd k, or of

$$\ln R_1 = -\sum_{n=1}^{\infty} R^n \frac{\cos (n \cos^{-1} \mu)}{n} \tag{150}$$

for even k.

Similarly, the results obtained in Sec. 4 where it was shown that a general solution of equation (1) may be obtained by means of (69) from a proper solution of (17) for $k = 1$, that is, from axially symmetric three-dimensional harmonics, can be carried over to obtain harmonics in odd numbers of dimensions possessing the type of symmetry indicated by equation (20), as follows:

$$\varphi_{2k+1}(R, \mu) = \left(\frac{1}{R}\frac{\partial}{\partial \mu}\right)^k \varphi_3(R, \mu), \tag{151}$$

where the notation is similar to that of equation (69). The proof follows along the lines of equation (69). Likewise, for even numbers of dimensions the following relation holds:

$$\varphi_{2k}(R, \mu) = \left(\frac{1}{R}\frac{\partial}{\partial \mu}\right)^k \varphi_2(R, \mu), \tag{152}$$

where φ_2 is a harmonic function in the variables $x = R\mu$, $r = R\sqrt{1 - \mu^2}$. Since a (real) two-dimensional harmonic function φ_2 can be written in the form

$$\varphi_2 = \text{Re}\ \{f[\text{R}(\mu + i\sqrt{1 - \mu^2})]\} \tag{153}$$

where f is an analytic function of its argument, equation (152) reduces to

$$\begin{aligned} \varphi_{2k}(R, \mu) &= \text{Re} \left(\frac{1}{R}\frac{\partial}{\partial \mu}\right)^k f[R(\mu + i\sqrt{1 - \mu^2})] \\ &= \text{Re} \left\{\left(\frac{1}{R}\frac{\partial}{\sin\theta\, \partial\theta}\right)^k [f(R\, e^{i\theta})]\right\}. \end{aligned} \tag{154}$$

This, unlike (148) and (149), involves *no integrations* of any kind, but purely *differentiations*. It has thus been shown that even-dimensional harmonic functions possessing the symmetry (20) can be expressed in terms of a single analytic function of a complex variable in the form (154) involving only differentiations. In particular, for four-dimensional harmonics, one obtains

$$\varphi_4(R, \mu) = -\text{Im}\left[\frac{f'(R\, e^{i\theta})e^{i\theta}}{\sin\theta}\right]. \tag{155}$$

Thus far k has been considered to be a positive integer only. However, equations (17) to (19) may also be considered for noninteger values of k, and some of the above solutions, such as the series (146) and the integral representation (148) may still be used to furnish solutions of (17) for such values of k. Purely by analogy to positive integer k, one speaks of equation (17) as representing the Laplace equation in a fractional (or negative and even complex) number of dimensions. For properly multiple-valued functions f, even the solution (149) may be generalized to fractional k. A similar statement applies to "spherical harmonics" resulting from the expansion of R_1^{-k} in powers of R and to expansions of solutions of (17) in terms of these "harmonics." Equations similar to (151) can be used to relate solutions of (17) for values $k = k' + 2n$ to solutions of the same equation for $k = k'$.

Reference should be made to Bateman [4] and Bers and Gelbart [6]. In [4], page 408, an integral representation of solutions of (17), essentially equivalent to (148), is given; due to what is probably a typographical error, the constant $C_{(k-1)/2}$ is inadvertently replaced by π in [4]. In [6] the authors treat a set of equations even more general than (19), in which (in the present notation) r^k is replaced by an arbitrary function of r; they introduce a generalization of analytic functions and show that $\varphi + i\psi$ constitutes such a generalization. By extending the notion of the integral of an analytic function, they are able to obtain further solutions of (19) from a given set of solutions. It may be shown that this integral method, when applied to (19), is equivalent to replacement of φ_0 in (146) or in (148) by its integral $\int \varphi_0(x)\,dx$. The set of solutions obtained in [6] for equations (19) by successive integrations, starting with $\varphi = 1$, is merely the set of spherical harmonics of integral degree.[1]

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[1] [6] also contains the Bateman integral formula with the same numerical error as in [4].

STRESS ANALYSIS IN ELASTIC-PLASTIC STRUCTURES

BY

W. PRAGER AND P. S. SYMONDS

1. Introduction. In many problems of structural design, the designer knows fairly well the extreme values of the individual load components but has no a priori knowledge of the manner in which these loads combine or follow each other. For a vehicular bridge, for instance, individual dynamic loads, wind loads, snow loads, etc., may have well-specified *extreme values*, while the order of their application and their precise values at a given instant are not known beforehand.

If the structure is assumed to behave *elastically*, this uncertainty regarding the loads does not create any difficulty: the effects of the various types of loads may be studied separately, and the most unfavorable combination of loads may then be determined for each element of the structure. This procedure is based on two important features of the mathematical theory of elasticity: the linear character of the theory and the fact that it disregards all time effects. Thanks to the linearity of the theory, the effects of the various loads may be superimposed, and because time effects are disregarded, the effect of a certain load at a certain instant does not depend on the length of the period during which the load has already been acting on the structure. Thus, the combined effect of several loads acting simultaneously is independent of the order in which these loads have been brought upon the structure and of the time intervals through which they have already been acting on the structure.

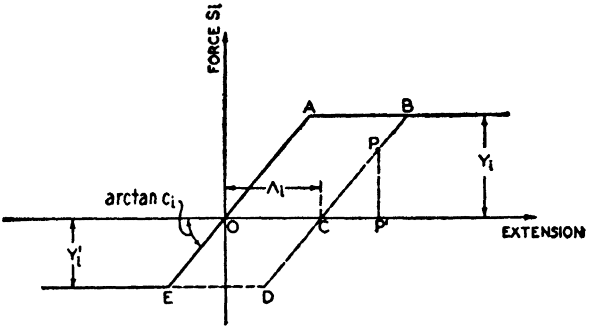
In a *linear* theory of visco-elasticity, the effects of several loads may still be superimposed, but since each of these effects depends on time, the combined effect of the loads is no longer independent of the history of loading. Any theory of elastic-plastic behavior, on the other hand, must be based on a nonlinear stress-strain law, even if all time effects are neglected. Thus, the aforementioned uncertainty regarding the sequence and detailed variation of the loads will create difficulties when visco-elastic-plastic structures are considered, because it is then no longer possible to superimpose the effects of the various types of loads in the same unrestricted manner in which this is possible for an elastic structure. In fact, the question arises whether this uncertainty does not force the designer to stress his structure within the elastic limit, thus forfeiting the often considerable advantages which he might derive from the ductility of structural materials.

In the present paper, this question is discussed for indeterminate elastic-plastic trusses. This particularly simple type of elastic-plastic structure lends itself well to such a discussion, and the concepts developed in this paper have already been successfully applied to continuous beams and frames [1].

2. Basic concepts. For simplicity, only *plane* trusses will be considered in the following. Any movable support will be replaced by a supporting bar which connects the supported joint to the ground and is perpendicular to the displacement allowed by the movable support. Let b denote the total number of bars including the supporting bars, j the total number of joints, and k the number of joints located in the ground. Since the truss is supposed to be statically indeterminate,

$$b > 2(j - k). \tag{1}$$

The mechanical behavior of the ith bar in simple tension or compression is characterized by three constants: the spring constant c_i, and the yield forces Y_i and Y'_i. The spring constant represents the force per unit elastic elongation; the bar yields in tension under the constant force Y_i and in compression under the constant force $-Y'_i$ (see the force-elongation diagram, Fig. 1). When the bar is loaded in tension, it first behaves elastically (OA) and then flows under constant force (AB). If the bar is unloaded after a point such as B has been reached, the line BC representing force versus elongation during the unloading process is straight and parallel to the elastic line OA. After complete unloading, the permanent elongation OC is observed. If the bar is then loaded in compression, the force-elongation diagram follows the continuation CD of the unloading line BC until the force reaches the value $-Y'_i$; thereafter the bar flows in compression under the constant force $-Y'_i$. The shape of the force-elongation diagram is not supposed to depend on the speed of the deformation.



REDUCED FORCE: $s_i = S_i/\sqrt{c_i}$

REDUCED YIELD FORCES: $y_i = Y_i/\sqrt{c_i}$, $y'_i = Y'_i/\sqrt{c_i}$

REDUCED PERMANENT EXTENSION: $\lambda_i = \Lambda_i \sqrt{c_i}$

REDUCED TOTAL EXTENSION: $s_i + \lambda_i$

FIG. 1

Consider a generic point P on this force-extension diagram (Fig. 1). The total extension OP' may be decomposed into the *permanent extension* OC and the *elastic extension* CP', the latter being that part of the total extension which disappears upon unloading from P. Let S_i denote the force and Λ_i the permanent extension. Since the elastic extension is S_i/c_i, the total extension equals $(S_i/c_i) + \Lambda_i$.

In the following, it will prove convenient to introduce the *reduced forces*,

$$s_i = \frac{S_i}{\sqrt{c_i}}, \tag{2}$$

the *reduced yield* forces,

$$y_i = \frac{Y_i}{\sqrt{c_i}}, \qquad y_i' = \frac{Y_i'}{\sqrt{c_i}}, \tag{3}$$

and the *reduced permanent extension*,

$$\lambda_i = \Lambda_i\sqrt{c_i}. \tag{4}$$

If the elastic and total extensions, too, are reduced by multiplying them by $\sqrt{c_i}$, the *reduced elastic extension* is found to equal s_i, and the *reduced total extension* is found to equal $s_i + \lambda_i$.

A state of stress of the elastic-plastic truss may be specified by the values of the b parameters s_i and represented by the point with the rectangular Cartesian coordinates s_i in a b-dimensional Euclidean space. This space will be called the *stress space*; the point representing the state of stress will be called the *stress point*, and the radius vector $\mathbf{s}$ from the origin to this point the *stress vector*. The yield condition requires that

$$-y_1' \leqq s_i \leqq y_i. \tag{5}$$

The stress point can, therefore, not lie outside the *yield parallelepiped* which is defined by the inequalities (5) written for all b bars of the truss. It is important to note that this domain to which the stress point is restricted by the yield condition is *convex.*

A state of permanent strain ("set") in the elastic-plastic truss may be specified by the values of the b parameters λ_i. Since the λ_i and s_i have the same dimension ($\sqrt{\text{force}\cdot\text{length}}$), it is possible to represent the state of permanent strain, too, in the stress space. The point with the coordinates $-\lambda_i$ will be called the *set point* and the radius vector $-\boldsymbol{\lambda}$ from the origin to this point the *set vector.* Since the reduced total extension equals $s_i + \lambda_i$, the state of total strain in the truss is represented by the vector from the set point to the stress point.

At each of the $j - k$ joints which are not located in the ground, two equations of equilibrium must be satisfied by the loads applied to this joint and the forces in the bars which meet there. Each of these $2(j - k)$ equations of equilibrium may be written in such a manner that the left-hand side is a linear form of the reduced forces s_i while the right-hand side is proportional to a load component. Moreover, the coefficients of the reduced forces s_i on the left-hand side depend only on the geometry of the structure and not on the intensity of the loads. Thus, each equation of equilibrium is represented by a hyperplane in stress space, and this hyperplane undergoes a translation as the load component on the right-hand side of the equation varies. The intersection of the $2(j - k)$ hyperplanes corresponding to given intensities of the loads is a linear subspace of $b - 2(j - k)$ dimensions called the *subspace of equilibrium* for the given loads. Depending on the history of loading, the structure can assume various states of stress under these loads. The corresponding stress points, however, must all lie in the subspace of equilibrium for these loads.

A state of deformation of the truss may be specified by giving the values of $2(j - k)$ displacement components (two for each of the $j - k$ joints which are not located in the ground). For the *small* displacements considered here, the extensions of the b bars may be expressed as linear forms in these $2(j - k)$ displacement components. On account of (1), the displacement components may be eliminated between these equations for the bar extensions. Thus,

$$b - 2(j - k)$$

linear *homogeneous* equations are obtained which the extensions must satisfy. In the following, these equations, written in terms of the reduced extensions, will be called the equations of *compatibility.*

In stress space, the reduced extensions are the components of the vector from the set point to the stress point. Each of the equations of compatibility may therefore be regarded as the equation of a hyperplane through the set point; the intersection of these hyperplanes is a $2(j - k)$-dimensional linear subspace through the set point, and the stress point must lie in this *subspace of compatibility.*

As the permanent extensions (and, hence, the position of the set point) vary, the subspace of compatibility undergoes a translation. In particular, as long as the truss is stressed within the elastic range, the set point coincides with the origin. The subspace of compatibility through the origin must therefore contain the stress points of all purely elastic states of stress in the truss. This $2(j - k)$-dimensional linear subspace will therefore be called the *subspace of elastic states.*

The subspace of equilibrium corresponding to a given set of loads and the subspace of elastic states intersect in a single point, the stress point representing the uniquely determined elastic state of stress under the given loads. If this stress point lies outside the yield parallelepiped, this elastic state of stress cannot be reached, and permanent extension occurs in at least one bar. If the permanent extensions and, hence, the set point are known, the stress point is uniquely determined as the intersection of the subspace of equilibrium and the subspace of compatibility through the given point.

Let $\mathbf{s}$ denote the radius vector of a point in stress space which lies on the surface of the yield parallelepiped but outside the subspace of elastic states. If the stress point can reach the position $\mathbf{s}$ at all, it certainly can do so only after a permanent extension has occurred in at least one bar. Suppose that the stress point has reached the position $\mathbf{s}$. A subsequent variation of the loads will be said to constitute unloading, if it does not produce any further permanent extension. Since the set point thus remains fixed during unloading, the stress point must move in the subspace of compatibility through the given set point. As the loads are gradually reduced, the subspace of equilibrium undergoes a translation towards the origin. Finally, when the loads have all been removed, the subspace of equilibrium passes through the origin. This particular subspace of equilibrium will be called the *subspace of residual states.* Indeed, if no plastic flow occurs during the unloading process, the residual state of stress obtained after complete removal of the loads is represented by the point of intersection of the subspace of compatibility through the given set point and the subspace of residual states. If this *residual stress point* lies outside the yield parallelepiped; it represents a fictitious state of residual stress which cannot be reached; in this case plastic flow will occur during the later stages of the unloading process.

3. Orthogonality relations. In the following section, the minimum principles which govern the stress analysis in elastic-plastic trusses will be expressed in the geometrical language of stress space. Since these minimum principles are not so widely known as they merit, and since their derivation can be made very brief at least in the case of elastic-plastic trusses, they will be rederived rather than taken for granted.

The principal ingredient in the proof of each of the minimum principles for elastic-plastic trusses is the orthogonality of the subspace of equilibrium and the subspace of compatibility. This orthogonality is most readily deduced from the principle of minimum strain energy for *elastic* trusses. According to this principle, the actual state of stress under given loads furnishes a smaller value of the strain energy than any other state of stress which is in equilibrium with the same loads.

For an *elastic* truss, the vector $\mathbf{s}$ in stress space represents the reduced forces and also the reduced extensions, and the square of the length of $\mathbf{s}$ is easily seen to equal twice the strain energy. The principle of minimum strain energy stipulates therefore that the elastic stress point is nearer to the origin than any other point of the subspace of equilibrium for the given loads. This means that the stress vector $\mathbf{s}$ is orthogonal to the subspace of equilibrium for the loads which produce the stresses represented by $\mathbf{s}$. As the loads are varied, the subspace of equilibrium undergoes a translation while the end point of the vector $\mathbf{s}$ describes the subspace of compatibility which, for an elastic truss, passes through the origin. Thus, the two subspaces are orthogonal.

For an *elastic-plastic* truss, the situation is the same as long as the truss is stressed within the elastic range. Thus, the subspace of equilibrium is orthogonal to the subspace of elastic states. The latter, however, is nothing but the subspace of compatibility through the origin. As permanent strains arise, the subspace of compatibility undergoes a *translation* so as to pass always through the set point. Accordingly, the subspace of compatibility is always orthogonal to the subspace of equilibrium. To obtain an equation which expresses this statement, let $\mathbf{s}$ represent the actual state of stress which is reached under a given set of loads after the permanent strains represented by $\boldsymbol{\lambda}$ have developed, and let $\mathbf{s}'$ be another state of stress which is in equilibrium with the same set of loads. The vector $\mathbf{s}' - \mathbf{s}$ is then contained in the subspace of equilibrium and the vector $\mathbf{s} + \boldsymbol{\lambda}$ in the subspace of compatibility. The orthogonality of these two subspaces requires that

$$(\mathbf{s}' - \mathbf{s}) \cdot (\mathbf{s} + \boldsymbol{\lambda}) = 0, \tag{6}$$

where the dot indicates scalar multiplication.

Consider now small changes of the loads which will produce changes of stress and permanent strain represented by $d\mathbf{s}$ and $d\boldsymbol{\lambda}$, respectively. Furthermore, let $d\mathbf{s}'$ represent another set of changes of stress which are in equilibrium with the same changes of the loads. As these changes occur, both the subspace of equilibrium and the subspace of compatibility undergo translations. Thus, the vector $\mathbf{s} + d\mathbf{s} + \boldsymbol{\lambda} + d\boldsymbol{\lambda}$ which lies in the new subspace of compatibility is orthogonal to the vector $\mathbf{s}' - \mathbf{s}$ which lies in the old subspace of equilibrium, and the vector $\mathbf{s}' + d\mathbf{s}' - \mathbf{s} - d\mathbf{s}$ in the new subspace of equilibrium is orthogonal to the vector $\mathbf{s} + \boldsymbol{\lambda}$ in the old subspace of compatibility. Combining the equations which express these facts with equation (6), one obtains

$$(\mathbf{s}' - \mathbf{s}) \cdot (d\mathbf{s} + d\boldsymbol{\lambda}) = 0 \tag{7}$$

and

$$(d\mathbf{s}' - d\mathbf{s}) \cdot (\mathbf{s} + \boldsymbol{\lambda}) = 0. \tag{8}$$

Finally, $\mathbf{s} + d\mathbf{s} + \boldsymbol{\lambda} + d\boldsymbol{\lambda}$ in the new subspace of compatibility is orthogonal to $\mathbf{s}' + d\mathbf{s}' - \mathbf{s} - d\mathbf{s}$ in the new subspace of equilibrium. Combining the equation expressing this orthogonality with equations (6) to (8), one finds

$$(d\mathbf{s}' - d\mathbf{s}) \cdot (d\mathbf{s} + d\boldsymbol{\lambda}) = 0. \tag{9}$$

For equations (6) to (9) to be valid, the state of stress $\mathbf{s}'$ must be in equilibrium with the given loads. It will prove useful in the following to restrict the choice of $\mathbf{s}'$ still further by demanding that this state of stress must not violate the yield conditions of the bars of the truss. In other terms, the point with the radius vector $\mathbf{s}'$ must not lie outside the yield parallelepiped.

4. Minimum principles. From the discussion of Sec. 2, it follows that, for a given set point, the stress point may be characterized as that point of the subspace of equilibrium which is nearest to the set point. In other terms, for a given state of permanent deformation $\boldsymbol{\lambda}$, the actual state of stress $\mathbf{s}$ corresponds to a smaller value of $(\mathbf{s} + \boldsymbol{\lambda})^2$ than any other state of stress which is in equilibrium with the given loads. Now, $(\mathbf{s} + \boldsymbol{\lambda})^2$ equals twice the strain energy computed from the total extensions as if these had occurred within the elastic range. In the following, this energy will be called the *fictitious strain energy corresponding to the total extensions.*

Among all states of stress which are in equilibrium with the given loads, the one which will actually occur for given values of the permanent extensions is thus seen *to minimize the fictitious strain energy corresponding to the total extensions.* This is the special form which G. Colonnetti's minimum principle [2] assumes for elastic-plastic trusses. All that is needed to obtain the general form of this principle, valid for all types of elastic-plastic structures, is to replace the term "extension" in the preceding statement of the principle by the more general term "strain." While Colonnetti established only a relative minimum, the preceding discussion shows that, for elastic-plastic trusses, one has to do with an absolute minimum. This result is easily extended to elastic-plastic structures in general.

At first glance, Colonnetti's principle seems to be open to two objections: On one hand, it specifies the stresses in an elastic-plastic structure under given loads in a manner which does not seem to reflect the history of loading. On the other hand, the practical value of the principle might be questioned because it seems to require that the permanent strains be known throughout the structure before the stresses can be obtained, and it is not clear how these permanent strains can be determined.

In fact, the latter objection furnishes the argument necessary to remove the first: It is precisely through the existing permanent strains that the history of loading is reflected in the determination of the stresses from the given load. Thus, Colonnetti's principle does not claim that the stresses in an elastic-plastic structure under given load are independent of the history of loading, but only that two loading processes which terminate in the same set of loads and the same set of permanent strains cannot be distinguished from each other by observing the subsequent mechanical behavior of the structure.

The second possible objection to Colonnetti's principle is more serious. For elastic-plastic trusses, however, it can be countered by the following argument: Within the elastic range, Colonnetti's principle reduces to the principle of minimum strain energy which is known to furnish unique stresses in all bars. As the loads, or some of them at least, are increased, some bars will begin to yield, and the principle of minimum strain energy must be replaced by Colonnetti's principle. Every time a bar begins to yield, the stress in this bar ceases to be an unknown since it will equal the known yield stress of this bar as long as the bar is yielding. On the other hand, the permanent extension of the bar enters as a new unknown. Every time a bar stops yielding, the permanent extension of this bar drops out as an unknown since it will remain constant as long as the bar does not yield, and the stress in the bar again becomes an unknown. Colonnetti's principle can be shown to furnish the necessary number of linear equations throughout each phase of the loading program. To obtain the stresses and permanent extensions in an elastic-plastic truss, however, it is necessary to follow the loading program phase by phase. For some other types of elastic-plastic structures, the second objection to Colonnetti's principle can be countered in a similar manner, but as far as the elastic-plastic continuum in general is concerned, this objection appears to be justified.

A minimum principle which is not open to this objection has recently been given by H. J. Greenberg [3]. With the notation introduced above, the special form which this principle assumes for elastic-plastic trusses may be proved as follows: For $ds' \neq ds$, the expression $(ds' - ds)^2$ is positive; hence

$$(ds' - ds)^2 - 2ds^2 > -2ds^2$$

or

$$ds'^2 - ds^2 > 2ds \cdot (ds' - ds).$$

With the aid of (9), this inequality may be written in the form

$$ds'^2 - ds^2 > -2d\lambda \cdot (ds' - ds). \tag{10}$$

Now,

$$d\lambda \cdot ds = 0, \tag{11}$$

because $d\lambda_i = 0$ as long as the ith bar does not yield, and $ds_i = 0$ when this bar is yielding. Moreover,

$$d\lambda \cdot ds' \leqq 0. \tag{12}$$

Indeed, if the ith bar is yielding in tension, $d\lambda_i > 0$, and a change of stress which is compatible with the yield condition satisfies $ds_i' \leqq 0$. Thus, $d\lambda_i ds_i' \leqq 0$. The same inequality is found to hold when the ith bar is yielding in compression. Finally, $d\lambda_i = 0$ when the ith bar does not yield.

With the aid of (11) and (12), the inequality (10) may be written in the following form

$$ds'^2 > ds^2. \tag{13}$$

In the geometrical terminology introduced above, this result may be expressed as follows: As the loads acting on an elastic-plastic truss are varied, the stress point describes the shortest path which is compatible with the conditions that, at any instant, the stress point must lie (1) within the yield parallelepiped and (2) in the subspace of equilibrium corresponding to the instantaneous loads. Alternatively, the inequality (13) may be expressed by stating that, *for a given change of loads, the elastic strain energy computed from the actual change of stress is smaller than that computed from any other change of stress which is compatible with the yield condition and in equilibrium with the given change of loads.* This is Greenberg's principle [3]. Involving only the change of stress corresponding to a given change of loads, it clearly exhibits the influence of the history of loading: As a rule, the state of stress associated with given loads can be obtained only by tracing the influence of each load increment.

It is worth noting, however, that under certain conditions the precise loading program may be irrelevant. Suppose, for instance, that no bar which has reached its yield stress some time during the loading process has ever been under a smaller stress thereafter. In this case, it can be shown that *the elastic strain energy computed from the actual stress is smaller than that computed from any other state of stress which is compatible with the yield condition and in equilibrium with* *the given loads.* This is the minimum principle of Haar and von Kármán [4]. Its proof is similar to that of Greenberg's principle. For $\mathbf{s}' \neq \mathbf{s}$,

$$(\mathbf{s}' - \mathbf{s})^2 > 0.$$

From this and equation (6), it follows that

$$\mathbf{s}'^2 - \mathbf{s}^2 > -2\boldsymbol{\lambda}\cdot(\mathbf{s}' - \mathbf{s}). \tag{14}$$

Now, $\lambda_i = 0$ as long as the ith bar has not reached its yield limit. Once this bar has reached its tensile yield limit, $\lambda_i > 0$ and $s_i = y_i$ for all later times and $s_i' \leqq y_i$, since the yield condition must not be violated. Similarly, once the ith bar has started to yield in compression, $\lambda_i > 0$, $s_i = -y_i'$, and $s_i' \geqq -y_i'$. In either case, $\lambda_i(s_i' - s_i) \leqq 0$. Thus, the inequality (14) reduces to

$$\mathbf{s}'^2 > \mathbf{s}^2. \tag{15}$$

This inequality expresses the minimum principle of Haar and von Kármán. In this form, the principle stipulates minimization of the elastic strain energy computed from the *actual* stresses. An alternative form of the principle is concerned with the elastic strain energy computed from the *residual* stresses. A state of stress $\mathbf{s}$ in the truss may be considered as the result of the superposition of (1) the residual state of stress $\mathbf{r}$ which would be obtained after complete unloading if no yielding occurred during unloading, and (2) the elastic state of stress $\mathbf{s}_0$ which would be produced by the actual loads in an indefinitely elastic truss. Since $\mathbf{r}$ is in the subspace of residual states, *i.e.*, in the subspace of equilibrium for vanishing loads, and $\mathbf{s}_0$ in the subspace of elastic states, *i.e.*, the subspace of compatibility for vanishing permanent strains, the vectors $\mathbf{r}$ and $\mathbf{s}_0$ are orthogonal. Thus, $\mathbf{s}^2 = \mathbf{r}^2 + \mathbf{s}_0^2$. Similarly, $\mathbf{s}'^2 = \mathbf{r}'^2 + \mathbf{s}_0^2$. Equation (15) therefore implies that, for $\mathbf{r}' \neq \mathbf{r}$,

$$\mathbf{r}'^2 > \mathbf{r}^2. \tag{16}$$

The elastic strain energy computed from the actual residual stresses is thus seen to be smaller than the elastic strain energy computed from the residual stresses associated with any other state of stress which is compatible with the yield condition and in equilibrium with the given loads. Like the principle of Haar and von Kármán, this principle is valid only as long as no unloading has occurred anywhere in the truss.

5. Limit design. Consider an elastic-plastic truss subjected to loads which vary with time in such a manner that the ratio of the intensities of any two loads remains constant (*proportional loading*). As the loads vary in this manner, the $[b - 2(j - k)]$-dimensional subspace of equilibrium sweeps a linear subspace of $b - 2(j - k) + 1$ dimensions which will be called the subspace of *proportional loading* corresponding to the given ratios between the loads.

Obviously, any process of proportional loading can be discussed within the appropriate subspace of proportional loading. The intersection of this subspace and the yield parallelepiped will be called the *yield domain* for the considered process of proportional loading. The yield domain is a *convex polyhedron* in the subspace of proportional loading, and the subspace of equilibrium corresponding to given load intensities is a *hyperplane*. When the load intensities are increased in proportion, the hyperplane of equilibrium undergoes a translation away from the origin. For sufficiently small loads, the truss behaves elastically and the stress point is the foot of the perpendicular dropped from the origin on to the hyperplane of equilibrium. As the loads are increased in proportion, the stress point will eventually reach a face of the yield domain and then, according to Greenberg's principle (see Sec. 4), proceed on this face in the direction perpendicular to its intersection with the hyperplane of equilibrium. Eventually, the stress point will reach an "edge" of this face (*i.e.*, its intersection with a neighboring face) and then either proceed on this edge or cross over to the neighboring face, depending on the orientation of the two faces and the hyperplane of equilibrium. In the first case, the bars corresponding to the two faces will both be yielding; in the second case, the bar corresponding to the face which was first reached by the stress point will stop yielding and the bar corresponding to the other face will start yielding. It is worth noting that this type of behavior cannot be ruled out by specifying proportional loading. In the preceding section the Haar-Kármán principle was proved under the assumption that no bar which has started yielding will ever stop yielding during the loading process under consideration. In view of what has just been shown, this condition is much more severe than might be expected at first glance.

If the stress point proceeds on the "edge" formed by the two faces, it must move normal to the intersection of this edge and the hyperplane of equilibrium. If, on the other hand, the stress point crosses over to the second face of the yield domain, it must move normal to the intersection of this face and the hyperplane of equilibrium. In each case, Greenberg's principle defines a unique direction of motion for the stress point. This continues to be the case as long as the hyperplane of equilibrium has not yet become a *supporting plane* of the yield domain. Any further proportional increase of the load intensities would displace the hyperplane of equilibrium to a position where it would have no points in common with the yield domain. In other words, any further proportional increase of the load intensities would call for stresses in excess of the yield stress, at least in some bars. Thus, the truss is unable to withstand any such increase of the loads: it has reached its *carrying capacity*.

For a given system of loads, the *safety factor* φ of the truss may be defined as that ratio in which the loads have to be increased in proportional loading if the truss is to reach its carrying capacity. To define the safety factor in geometrical terms, consider the hyperplane of equilibrium for the given loads and that supporting plane of the convex yield domain which lies on the same side of the origin and has the same orientation as this hyperplane of equilibrium. The desired safety factor then equals the ratio between the distances of the origin from the supporting plane and the hyperplane of equilibrium. From this geometric representation follows at once an important result which Feinberg [5] established in a different manner. Consider any state of stress $\mathbf{s}'$ which is compatible with the yield condition and in equilibrium with the given loads. Determine the largest factor ψ with which the stresses of this state can be multiplied without violating the yield condition. The safety factor φ of the truss for the given loads then is the largest value which ψ can assume as $\mathbf{s}'$ is varied. It, therefore, is "safe" to estimate the safety factor φ by the process which yields ψ. In particular, the truss has not reached its carrying capacity as long as it is possible to find a state of stress which is in equilibrium with the given loads and for which no bar has reached its yield limit.[1] This result is sometimes dubbed the principle of the inherent intelligence of the material; it is here seen to be an immediate consequence of the convexity of the yield domain.

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[1] Note that this state of stress need not satisfy the equations of compatibility of the theory of elasticity.

ELASTIC-PLASTIC STRAINING

BY

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This is an abstract of the author's Symposium address dealing with the implications of some numerical investigations conducted by the author together with D. N. de G. Allen and G. Vaisey. Two distinct problems were considered: two-dimensional elastic-plastic straining, and elastic-plastic torsion. Full accounts of these investigations, together with details of the relaxational methods used and numerous diagrams of the results, are contained in two papers that have been communicated by the author to the Royal Society of London and to the Quarterly Journal of Mechanics and Applied Mathematics [1, 2].

1. Elastic-plastic straining in systems of plane strain and plane stress. These problems were investigated under the hypothesis of the Prandtl-Reuss flow theory. According to this theory, material behaves elastically under stresses which do not satisfy the Mises-Hencky yield criterion, and flows without work hardening if this criterion is satisfied. During plastic flow, the strain is divided into two parts: an elastic part which is related to the stress by Hooke's law, and a plastic part which is equivoluminal. The increments of plastic principal shear strains are directly proportional to the principal shearing stresses and occur on the same planes.

The particular problems solved concerned notched bars in tension. Both semicircular and rectangular notches were investigated under conditions both of plane strain and of plane stress, making a total of four distinct solutions. It was found in each case that for a certain critical value of the tension on the bar plastic "enclaves" (plastic regions which are constrained by adjacent still-elastic regions so that large scale plastic flow is impossible) began to form. As the tension was increased, these enclaves grew larger. Eventually, enclaves from opposite sides of the bar would meet and large-scale flow would be permissible.

The shape of the enclaves was found to be quite different from the shape of plastic domains predicted when elastic strains are ignored.[1] Further, none of the solutions implied finite slipping at the boundary of the enclaves, whereas, this phenomenon is predicted when elastic strains are ignored. Therefore, although the neglecting of elastic strains may lead to mathematically simpler problems, the results are to be viewed with suspicion. Further, the investigations indicate that it is not necessary to make such additional assumptions, since relaxational methods appear adequate for solving two-dimensional problems with allowance for elastic strains.

The shape of the enclaves upon coming together from opposite sides of the bar provides a tentative explanation of the "cup and cone" type of fracture.

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[1] See [3], for instance.

2. Elastic-plastic torsion of a prismatic cylinder. In the case of torsion, the Prandtl-Reuss stress-strain law assumes a particularly simple form, since the directions of principal shearing stress do not alter *once plastic straining has begun.* The extent of the plastic strain at a point may be measured by the ratio of the magnitude of the shear-strain vector at any time to its magnitude at the time that point first became plastic.

The particular case of a prismatic bar with a hollow rectangular cross section having small fillets is solved for this ratio, using the stress solution of F. S. Shaw [4]. It is found that, even though plastic enclaves appear in the neighborhood of the fillets for a comparatively low torque, the plastic/elastic strain ratio remains less than 2 until the bar is almost entirely plastic.

The conclusions of the paper are compared with those obtained by P. G. Hodge, Jr. [5]. Hodge derived a method for obtaining directly the warping function in elastic-plastic torsion, rather than the strain ratio. Also, while Hodge's results can be applied to elastic-plastic torsion, he was primarily concerned with the "fully plastic state," a concept which the author finds less simple in regard to "hollow" than it is in regard to "solid" sections. The purpose of the present paper, on the other hand, is to illustrate the applicability of relaxational methods to problems in elastic-plastic torsion.

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A GRAPHICAL METHOD FOR SOLVING PROBLEMS IN PLANE PLASTICITY

BY

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1. Introduction. Many authors [1, 2] have investigated the distribution of stresses for a plastic body in a state of plane strain and satisfying the Saint-Venant–Mises yield condition. The principal studies in these investigations have been: (1) the geometry of the slip lines (characteristics), (2) the geometry of the net of principal stress directions, and (3) the integration of the characteristic system. The integration method can be easily applied to the case where one family of characteristics consists of straight lines and an explicit solution may be obtained. In other cases, the characteristic system is replaced by a system of difference equations and the Massau method of approximate integration is used. By combining both methods, W. W. Sokolovsky [3] has solved numerous specific problems.

We shall show that the equilibrium relations for the stresses and the above yield condition may be replaced by a system of first-order partial differential equations and a first-degree algebraic equation (the yield condition). Our system of first-order equations is not a characteristic system. In fact, our system is associated with the principal directions of a symmetric second-order tensor which is the gradient of a vector. For this reason, our system will be useful in those problems in which stress discontinuities[1] are not encountered. We shall show that, due to the geometric properties of our system, the whole problem of step-by-step integration of the difference equations reduces to the algebraic problem of solving two simultaneous algebraic equations at each step, one linear and one quadratic. Finally, it is shown that a simple graphical construction may be used to solve the above algebraic equations.

The method may be used to solve other hyperbolic differential equations encountered in plasticity such as in the Coulomb yield condition, etc.

2. The general theory. In this section, we denote the orthogonal Euclidean variables x, y of the plane by $x^\lambda (\lambda = 1, 2)$. Since we shall restrict ourselves to orthogonal Euclidean coordinates, no distinction exists between contravariant and covariant quantities. However, we shall introduce contravariant and covariant indices in order to use the summation convention.

For the case of plane strain, the plastic stress tensor, $\sigma^{\lambda\mu}$, in the absence of body forces is assumed to satisfy the equilibrium relations

$$\frac{\partial \sigma^{\lambda\mu}}{\partial x^\lambda} = 0, \qquad (\mu = 1, 2). \tag{1}$$

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[1] It should be noted that discontinuities in the stress derivatives are permitted. Further, stress discontinuities will never occur in the exterior problem but will always occur in the interior problem. Thus, our method is always applicable to the exterior problem but is applicable to that portion of the interior problem in which the stresses are continuous.

The additional relation which the stresses must satisfy is the yield condition. Although all our work through Sec. 4 may be extended to the more general Coulomb yield condition, we shall deal with the Saint-Venant–Mises yield condition. If σ_1, σ_2 denote the principal stresses ($\sigma_1 > \sigma_2$) and k is a constant, then the Saint-Venant–Mises condition is

$$\sigma_1 - \sigma_2 = k. \tag{2}$$

We note that if $\underset{1}{i}_\lambda$, $\underset{2}{i}_\lambda$ are unit vectors lying along the two orthogonal directions of principal stress, then the stress tensor may be written as

$$\sigma_{\lambda\mu} = \sigma_1 \underset{1}{i}_\lambda \underset{1}{i}_\mu + \sigma_2 \underset{2}{i}_\lambda \underset{2}{i}_\mu. \tag{3}$$

If we introduce the Airy stress function ϕ, then the solution of (1) is

$$\sigma^{\lambda\mu} = \epsilon^{\lambda\alpha}\epsilon^{\mu\beta} \frac{\partial^2 \phi}{\partial x^\alpha \, \partial x^\beta}. \tag{4}$$

where $\epsilon^{\lambda\mu}$ is the permutation tensor density (that is, $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = 1$). In order to develop our theory, we introduce the gradient vector

$$v_\lambda = \frac{\partial \phi}{\partial x^\lambda}, \tag{5}$$

and the symmetric tensor

$$h_{\lambda\mu} = \frac{\partial v_\lambda}{\partial x^\mu} = \frac{\partial^2 \phi}{\partial x^\lambda \, \partial x^\mu}. \tag{6}$$

Since $\epsilon^{\lambda\alpha}\epsilon_{\mu\alpha} = \delta^\lambda_\mu$, where δ^λ_μ is the Kronecker tensor, we may solve (4) for $h_{\lambda\mu}$. We obtain

$$h_{\alpha\beta} = \epsilon_{\lambda\alpha}\epsilon_{\mu\beta}\sigma^{\lambda\mu}. \tag{7}$$

If we assume that the principal directions $\underset{1}{i}_\lambda$, $\underset{2}{i}_\lambda$ are oriented in the same manner as the x^1, x^2 coordinate lines, then

$$\epsilon_{\lambda\mu} \underset{1}{i}^\mu = -\underset{2}{i}_\lambda, \qquad \epsilon_{\lambda\mu} \underset{2}{i}^\mu = \underset{1}{i}_\lambda. \tag{8}$$

Substituting (3) into (7) and using (8), we find

$$h_{\alpha\beta} = \sigma_2 \underset{1}{i}_\alpha \underset{1}{i}_\beta + \sigma_1 \underset{2}{i}_\alpha \underset{2}{i}_\beta. \tag{9}$$

Thus, *the principal directions of $h_{\alpha\beta}$ coincide with the principal directions of the stress tensor; the principal values of $h_{\alpha\beta}$ are equal to the interchanged principal stresses.*

The relations (4) through (9), excluding (8), are equivalent to the following system:

$$\underset{1}{i}^\mu \frac{\partial v_\lambda}{\partial x^\mu} = \sigma_2 \underset{1}{i}_\lambda, \tag{10}$$

$$\underset{2}{i}^{\mu}\frac{\partial v_\lambda}{\partial x^\mu} = \sigma_1 \underset{2}{i}_\lambda, \tag{11}$$

$$\frac{\partial v_\lambda}{\partial x^\mu} = \frac{\partial v_\mu}{\partial x^\lambda}, \tag{12}$$

$$\underset{1}{i}^{\lambda}\underset{2}{i}_\lambda = 0. \tag{13}$$

The condition (12) expresses the fact that v_λ is a gradient vector and (13) states that $\underset{1}{i}_\lambda$, $\underset{2}{i}_\lambda$ are orthogonal. *If the yield condition* (2) *is adjoined to this system, one obtains a system which is equivalent to the usual nonlinear second-order partial differential equation*[2] *for the Airy stress function.*

We now introduce a parametric net along the two families of principal directions $\underset{1}{i}_\lambda$, $\underset{2}{i}_\lambda$. If we let α, β denote, respectively, the curves along $\underset{1}{i}_\lambda$ and $\underset{2}{i}_\lambda$, then system (10) through (13) becomes

$$\frac{\partial v^\lambda}{\partial \alpha} = \sigma_2 \frac{\partial x^\lambda}{\partial \alpha}, \qquad (\lambda = 1, 2), \tag{14}$$

$$\frac{\partial v^\lambda}{\partial \beta} = \sigma_1 \frac{\partial x^\lambda}{\partial \beta}, \tag{15}$$

$$\frac{\partial x^\lambda}{\partial \alpha}\frac{\partial v_\lambda}{\partial \beta} = \frac{\partial x^\lambda}{\partial \beta}\frac{\partial v_\lambda}{\partial \alpha}, \tag{16}$$

$$g_{\lambda\mu}\frac{\partial x^\lambda}{\partial \alpha}\frac{\partial x^\mu}{\partial \beta} = 0, \tag{17}$$

where $g_{\lambda\mu}$ is the metric tensor of the system of orthogonal Euclidean coordinates x^λ. We note that equation (16) follows from (12) by forming the scalar product of both sides of the latter with $\underset{1}{i}^{\lambda}\underset{2}{i}^{\mu}$. However, equation (16) is superfluous, since by forming the scalar product of (14) with $\partial x^\lambda/\partial\beta$ and forming the scalar product of (15) with $\partial x^\lambda/\partial\alpha$ and using (17), we obtain

$$\frac{\partial x^\lambda}{\partial \alpha}\frac{\partial v_\lambda}{\partial \beta} = \frac{\partial x^\lambda}{\partial \beta}\frac{\partial v_\lambda}{\partial \alpha} = 0. \tag{18}$$

Thus, *equations* (14) *through* (17) *are equivalent to the simpler system* (14), (15), (17).

Equations (14), (15), (17) can be given a simple geometric interpretation; *i.e.*, they imply that *the x^λ-plane map of the principal directions, α = constant (or β = constant) is parallel to the v^λ-plane map of the same principal direction. Since the x^λ-plane map of the principal directions forms an orthogonal net, the v^λ-plane map of the principal directions furnishes an orthogonal net. Finally* (14), (15)

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[2] See [1], p. 638.

show that σ_2, σ_1, *respectively, are the scale factors which convert the net, consisting of* β = *constant*, α = *constant, respectively, from the* x^λ *plane to the* v^λ *plane.*

3. The boundary conditions. It is usually assumed that the applied force, along the known boundary curve C of the plane plastic region R, is known. If t^λ denotes the unit tangent vector to C and n^λ, the unit normal vector to C (the orientation of the pair t^λ, n^λ is assumed to be the same as that of the x^1, x^2 coordinate lines), then the equations

$$\sigma_{\lambda\mu} n^\lambda n^\mu = F_n, \qquad \sigma_{\lambda\mu} n^\lambda t^\mu = F_t, \tag{19}$$

determine the relation between the stress components $\sigma_{\lambda\mu}$ and the component of the applied force normal to the boundary F_n or tangent to the boundary F_t. Substituting (4) into (19) and using the relations (8) with $\underset{1}{i}^\lambda$ replaced by t^λ and $\underset{2}{i}^\lambda$ replaced by n^λ, we obtain

$$h^{\alpha\beta} t_\alpha t_\beta = F_n, \qquad h^{\alpha\beta} t_\alpha n_\beta = -F_t. \tag{20}$$

In view of (5), we may write equations (20) as

$$t_\lambda \frac{dv^\lambda}{ds_t} = F_n, \qquad n_\lambda \frac{dv^\lambda}{ds_t} = -F_t, \tag{21}$$

where ds_t represents a displacement along the positive direction (direction of t^λ) of C. Hence, we may write

$$\frac{dv_\lambda}{ds_t} = F_n t_\lambda - F_t n_\lambda. \tag{22}$$

Since the right-hand side of equation (22) consists of known functions and since C is known, we may easily determine v_λ (to within an arbitrary constant) along C. Equation (22) may be transformed into a form more suitable for computation by introducing the components of $t_\lambda \left(\frac{dx^1}{ds_t}, \frac{dx^2}{ds_t}\right)$ and $n_\lambda \left(-\frac{dx^2}{ds_t}, \frac{dx^1}{ds_t}\right)$. Then (22) furnishes

$$v^1 = \int_0^s F_n\, dx^1 + F_t\, dx^2, \tag{23}$$

$$v^2 = \int_0^s F_n\, dx^2 - F_t\, dx^1, \tag{24}$$

where s is a parameter along C.

4. The system of difference equations. To set up our system of difference equations, we consider a small triangle determined by points P_1, P_2, and P_3 of the x^λ plane. The point P_3 is to be determined by the intersection of the line β = constant through P_1 and the line α = constant through P_2. It is assumed that x^λ, v^λ are known at points P_1 and P_2. We denote these known quantities by x_1^λ, x_2^λ, v_1^λ, v_2^λ. In view of the geometric result given at the close of Sec. 3, the quantities v_3^λ can be obtained graphically when x_3^λ are known. In fact, we shall determine x_3^λ from a system of difference equations and then graphically determine v_3^λ in the v^λ plane. To facilitate the discussion, we replace x^1, x^2 by x, y and v^1, v^2 by u, v.

In terms of the above notation, equations (14), (15), (17), and (2) lead to the following system of difference equations:

$$u_3 - u_1 = \sigma_2(x_3 - x_1), \tag{25}$$
$$v_3 - v_1 = \sigma_2(y_3 - y_1), \tag{26}$$
$$u_3 - u_2 = \sigma_1(x_3 - x_2), \tag{27}$$
$$v_3 - v_2 = \sigma_1(y_3 - y_2), \tag{28}$$
$$(x_3 - x_2)(x_3 - x_1) + (y_3 - y_2)(y_3 - y_1) = 0, \tag{29}$$
$$\sigma_1 - \sigma_2 = k. \tag{30}$$

In order to simplify this system of six equations in the six unknowns u_3, v_3, x_3, y_3, σ_1, σ_2, we introduce, for the present, the slopes of the principal directions; *i.e.*, we write

$$\lambda = \frac{y_3 - y_1}{x_3 - x_1}, \qquad \mu = \frac{y_3 - y_2}{x_3 - x_2}. \tag{31}$$

We note that, if σ_2 is eliminated between equations (25) and (26), and σ_1 between (27) and (28), we obtain

$$\lambda(u_3 - u_1) - (v_3 - v_1) = 0, \tag{32}$$
$$\mu(u_3 - u_2) - (v_3 - v_2) = 0. \tag{33}$$

Solving equations (32) and (33) for u_3, we obtain

$$(\lambda - \mu)u_3 = \lambda u_1 - \mu u_2 - v_1 + v_2. \tag{34}$$

We now determine σ_2, σ_1 from (25), (27) and find

$$\sigma_2 = \frac{u_3 - u_1}{x_3 - x_1}, \qquad \sigma_1 = \frac{u_3 - u_2}{x_3 - x_2}. \tag{35}$$

Substituting (34) into (35), we obtain σ_2, σ_1 in terms of the known u's and v's and the unknown x_3, y_3,

$$\sigma_2 = \frac{\mu(u_1 - u_2) - (v_1 - v_2)}{(\lambda - \mu)(x_3 - x_1)}, \qquad \sigma_1 = \frac{\lambda(u_1 - u_2) - (v_1 - v_2)}{(\lambda - \mu)(x_3 - x_2)}. \tag{36}$$

Subtracting the equations in (36), we obtain

$$\sigma_1 - \sigma_2 = \frac{(u_1 - u_2)[\lambda(x_3 - x_1) - \mu(x_3 - x_2)] - (v_1 - v_2)(x_2 - x_1)}{(\lambda - \mu)(x_3 - x_1)(x_3 - x_2)}. \tag{37}$$

Replacing λ, μ by their values (31), we can write (37) as

$$\sigma_1 - \sigma_2 = \frac{(x_2 - x_1)(v_2 - v_1) - (y_2 - y_1)(u_2 - u_1)}{(x_3 - x_2)(y_3 - y_1) - (y_3 - y_2)(x_3 - x_1)}. \tag{38}$$

Substituting (38) into (30), we obtain the linear equation in x_3, y_3

$$(39)\quad k(y_2 - y_1)x_3 - k(x_2 - x_1)y_3 + k(y_1x_2 - y_2x_1) + (u_2 - u_1)(y_2 - y_1) - (v_2 - v_1)(x_2 - x_1) = 0.$$

Although (39) was obtained under the assumption, $x_3 \neq x_1$, $x_3 \neq x_2$, this equation is valid in the cases, $x_3 = x_1$, or $x_3 = x_2$. That is, if one works with y_i's [see (35)], then the relation (39) is obtained. The locus (29) represents a circle for which the segment joining (x_1, y_1), (x_2, y_2) is a diameter. Further, the line (39) is parallel to the line joining (x_1, y_1), (x_2, y_2). Hence, relations (29) and (39) determine two pairs of values for (x_3, y_3).

It is well known [3] that there exist two solutions for given boundary data F_n, F_t. In fact, from the well-known equation

$$(40)\qquad \sigma_{xy} = \frac{k}{2}\sin 2\alpha,$$

where α is the angle between the positive x axis and a family of principal directions, we obtain the equivalent relation

$$(41)\qquad \sigma_{\lambda\mu}t^\lambda n^\mu = F_t = \frac{k}{2}\sin 2\alpha,$$

where, now, α is the angle between the t^λ direction and a family of principal directions. From (41), we see that if F_t is known along C, then there exist two permissible values of α, that is, α_0 and $\pi/2 - \alpha_0$. But the intersections of the loci (29) and (39) determine exactly two such angles. Since the principal directions vary in a continuous manner, this continuity may be used to determine the required α_0. This rule will be satisfactory if α_0 is considerably less (in comparison with the allowable error in angle) than $\pi/4$. If α_0 is sufficiently close to $\pi/4$, this rule will break down. In that case, it is desirable to consider α_0 as $\pi/4$. The line element joining points (x_1, y_1), (x_2, y_2) is then a slip line. Further, the conditions $\sigma_1 - \sigma_2 = k$, $k > 0$, provide additional checks on the work. Finally, the equiareal property furnishes an important check.[3]

5. Procedure in solving boundary value problems. In order to carry out the graphical procedure for constructing the set of principal stress directions within the interior (or exterior) of any curve C, two cases have to be considered: (1) C, or a portion of C, is tangent to a principal direction; (2) C is not tangent to a principal direction. Cases 1 and 2 both arise in the computation. Usually, case 2 arises. However, if a closer approximation to the net is desired, it may be advisable to work along a principal direction.

First, we treat case 1. If this case occurs along the curve C, $\alpha = 0$, or $\alpha = \pi/2$. Returning to the notation of Sec. 4, the above implies that the linear element connecting points P_1 and P_3 is orthogonal to the linear element connecting P_1 and P_2. Hence, point P_1 plays the same role as point P_3 in the previous calculations (25) through (30). Interchanging the indices 1 and 3 in these equations, we see that (27) determines σ_1. Hence, (30) furnishes σ_2. Corresponding to some choice of point P_3 in the x^λ plane, equation (25) determines the map of P_3 in the v^λ plane. With the aid of a simple nomographic chart (involving only division, subtraction, and multiplication), all the computations may be performed graphically.

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[3] The author is indebted to Professor W. Prager for this remark.

We now come to case 2. In order to treat this case graphically, we introduce the vector from point P_1 to P_2 of the x^λ plane and denote it by $\vec{d}_{12}$ and its magnitude d_{12} (Fig. 1). Similarly, we define the vectors $\vec{d}_{13}$ (from point P_1 to P_3 and $\vec{d}_{23}$ (from point P_2 to P_3). The corresponding vectors of the v^λ plane, we denote by $\vec{e}_{12}$, $\vec{e}_{13}$, $\vec{e}_{23}$. From equations (38) and (30), we find

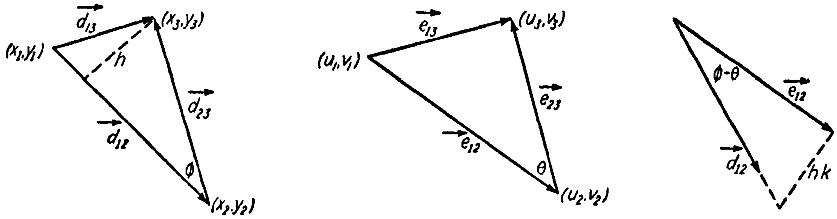

FIG. 1

$$\vec{d}_{12} \times \vec{e}_{12} = k\,\vec{d}_{23} \times \vec{d}_{13}. \tag{42}$$

Since the vectors $\vec{d}_{13}$, $\vec{e}_{13}$, and $\vec{d}_{23}$, $\vec{e}_{23}$ are parallel, we see from Fig. 1 that equation (42) may be written as

$$d_{12}e_{12} \sin(\phi - \theta) = kd_{23}\,d_{13}. \tag{43}$$

From the figure, we find

$$d_{23} = d_{12}\cos\phi, \qquad d_{13}\cos\phi = h. \tag{44}$$

Hence (43) becomes

$$kh = e_{12}\sin(\phi - \theta). \tag{45}$$

Thus, in order to determine the intersections of the line (39) and the circle (29) we proceed as follows: *Determine the projection of the line segment connecting points P_1 and P_2 in the v^λ plane onto the perpendicular to the line connecting points P_1 and P_2 in the x^λ plane; divide this projection by k; draw a line parallel to the line connecting points P_1 and P_2 of the x^λ plane and at this distance from the latter; find where this line intersects the circle.*

It should be noted that the introduction of the dimensionless quantities σ_1/k, σ_2/k, F_t/k will further simplify the above construction; *i.e.*, use of such quantities implies that $k = 1$ in the previous formulas.

6. Graphical results. The previously outlined theory was applied to the following problem: To determine the net of principal directions when constant normal and shear forces are applied to the interior circular boundary of a plane infinite sheet.

The xy plane map of boundary is

$$x = a \cos \theta, \qquad y = a \sin \theta. \tag{46}$$

The following values were chosen for the applied forces

$$-F_n = F_t = p = 0.45. \tag{47}$$

These are evidently consistent with the yield condition requirement that $| F_t | < 0.5$ for $k = 1$. From equations (23) and (24) we obtain the uv-plane map of (46),

$$u = ap\sqrt{2} \sin (\theta + \phi), \qquad v = ap\sqrt{2} \cos (\theta + \phi), \tag{48}$$

where

$$p\sqrt{2} = 0.637, \qquad \phi = 135°. \tag{49}$$

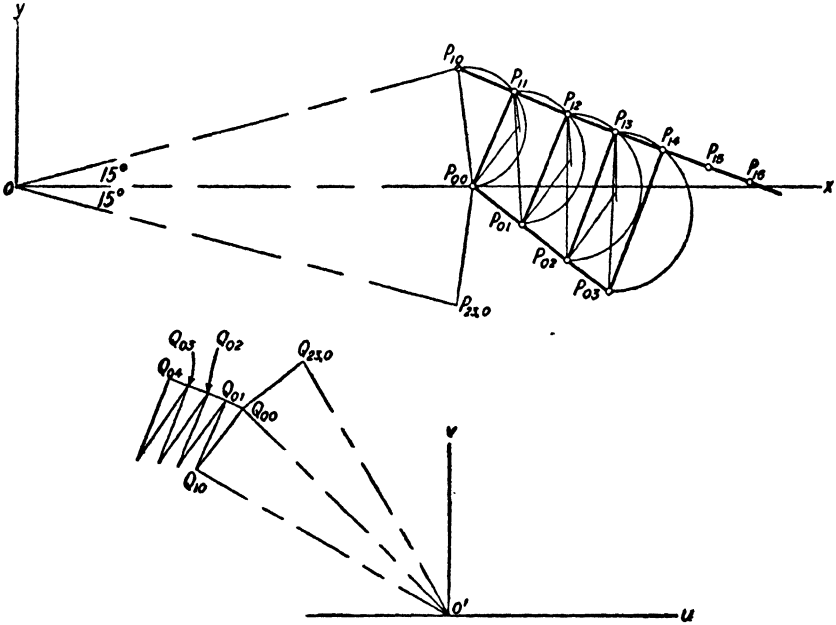

FIG. 2

In Fig. 2 the details of the construction for one element of the boundary are shown. A circle of 5 in. radius in the xy plane has been divided into arcs of 15° and chords (such as $P_{00}P_{10}$) formed. The map of $P_{00}P_{10}$ in the uv plane is $Q_{00}Q_{10}$ and is obtained through use of equations (48). A semicircle is described with $P_{00}P_{10}$ as diameter. Through the point P_{00}, a line element parallel to $Q_{00}Q_{10}$ and of equal length is drawn. Then, through the end point of this line segment, a line is drawn parallel to $P_{00}P_{10}$. The intersection of this latter line with the semicircle determines two points, one of which is P_{11}. Thus, $P_{10}P_{11}$ and $P_{00}P_{11}$ are two elements of the net of principal directions. The maps of these elements in the *uv* plane were obtained by drawing $Q_{00}Q_{01}$ parallel to $P_{10}P_{11}$, and $Q_{10}Q_{01}$ parallel to $P_{00}P_{11}$. Because of the circular symmetry, the point P_{01} can be easily obtained. For arbitrary contours, the determination of P_{01} requires a construction similar to that for P_{11} but involving $Q_{00}Q_{23,0}$ and $P_{00}P_{23,0}$. By proceeding in this manner, the four elements $P_{10}P_{11}$, $P_{11}P_{12}$, $P_{12}P_{13}$, $P_{13}P_{14}$ of one curve of the net of principal directions were obtained. The elements of the curves orthogonal to $P_{10}P_{14}$ are $P_{00}P_{11}$, $P_{01}P_{12}$, $P_{02}P_{13}$, $P_{03}P_{14}$. It should be noted that the principal stresses at P_{11} can be determined by finding the ratios, $Q_{01}Q_{00}:P_{10}P_{11}$, $Q_{01}Q_{10}:P_{11}P_{00}$.

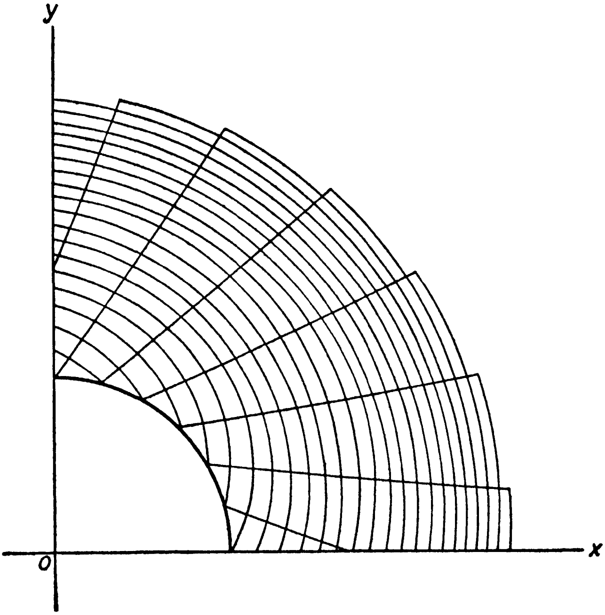

FIG. 3

Figure 3 furnishes a picture of the net of principal directions. Twenty steps of the type indicated in Fig. 2 were used and then the scale of Fig. 2 was reduced. In Fig. 3 the members of one family of principal directions appear to be straight lines. However, in the original drawing before reduction, some curvature was apparent. In view of the circular symmetry of the construction, errors in only the radial direction can be discussed. The angle of intersection of the two families of principal directions should be 90°. By close measurement, it appears that the error in intersection angle is uniform in the radial direction and is $7\frac{1}{2}°$. This is the error made in replacing the tangent line element to the circle by the chord. Therefore, it appears that this error may be eliminated by approximating the circle (in the xy plane) by tangent line elements and choosing P_{10} at a point of tangency.

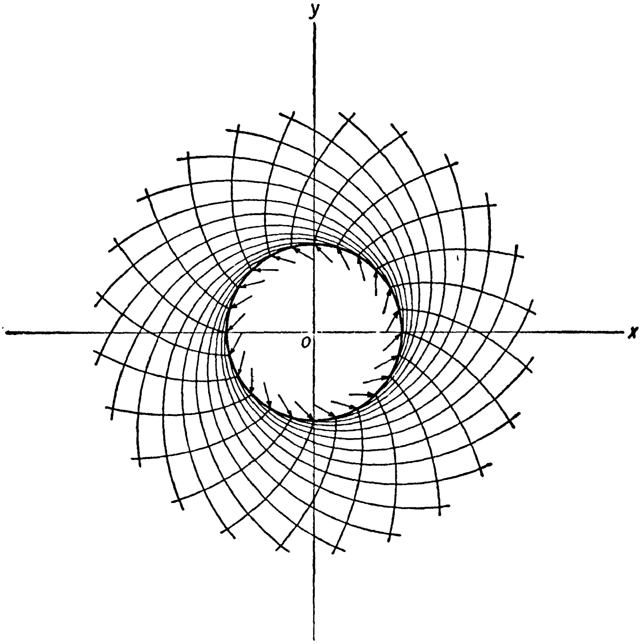

FIG. 4

Figure 4 shows the orthogonal families of characteristics which were constructed by graphical interpolation from Fig. 2. These curves are very similar to the corresponding curves in W. W. Sokolovsky's text.[4]

An attempt was made to treat the problem of a uniform normal force applied to an interior elliptic boundary. Owing to lack of symmetry, the various steps must be carried out in detail. Because of lack of time, this problem was not completed. It should be noted that the initial steps in this construction must be carried out by the first procedure outlined in Sec. 5.

In conclusion, the author wishes to thank the Horace H. Rackham School of the University of Michigan, for a Faculty Research Fund grant which made the construction of the graphs possible. Further, the author is indebted to Mr. W. Wilson for the graphical work.

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[4] See reference [3], p. 124.

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ON STRESS DISCONTINUITIES IN PLANE PLASTIC FLOW[1]

BY

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Summary. Quasi-static plastic flow in plane strain is considered, adopting the Saint-Venant theory of an ideal plastic material subjected to strains large in comparison with elastic strains. Prager has shown that the consideration of surfaces of stress discontinuity, between regions of continuous stress distribution, enables certain stress boundary-value problems to be solved for which continuous solutions cannot be found. The use of this concept, however, provides a variety of solutions for many stress boundary-value problems, all of which satisfy the equations of equilibrium and the condition for plastic flow. In this paper, the conditions along a discontinuity are examined, and it is shown that restrictions are imposed on the equations governing the velocities of flow, which must be satisfied in conjunction with the stress conditions. Application to the problem of plastic flow in a wedge with uniform pressure on one face produces a unique set of solutions. A surface of stress discontinuity is necessary for acute-angled wedges, while the discontinuous stress solution violates velocity conditions for obtuse-angled wedges for which a continuous solution exists.

The additional conditions are based on the deduction that the surface of stress discontinuity is an elastic membrane, and the Saint-Venant theory must be extended to take into consideration elastic strains in the membrane, although, in general, elastic material is considered rigid in this theory. In the membrane, elastic strains permit large changes of curvature which are associated with large plastic strains in the surrounding material.

The additional conditions based on the velocity equations confirm the present trend of thought that the so-called static determinacy of the stress equations is misleading in the application of the theory of plastic flow in plane strain. This general question is discussed briefly.

The validity of a method of using fictitious stress discontinuities to obtain approximate complete solutions is also discussed.

1. The theory of plastic flow in plane strain. We shall consider plastic flow in plane strain of an ideal plastic material which does not exhibit strain hardening. This is a satisfactory approximation for many materials, for example, mild steel during the lower yield-point elongation, work-hardened materials which have a sharp yield point; and even for materials like lead which have a gradual transition to plastic flow, a satisfactory analysis can be obtained by choosing an appropriate average value for the flow stress [1].

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[1] The results presented in this paper were obtained in the course of research conducted under Contract N7onr-358 sponsored jointly by the Office of Naval Research and the Bureau of Ships.

We shall consider problems where the material flowing plastically is not entirely surrounded by constraining boundaries or by elastic material, so that strains which are large compared with elastic strains occur. For the analysis of such problems, it is permissible to neglect strains of the order of elastic strains, and so to assume elastic material to be rigid. This assumption covers a wide class of problems, for the yield-point elongation of most metals is of the order 10^{-3}, so that this assumption would be reasonable even for strains as small as 10^{-2}. In many metal-forming problems strains of the order unity occur. For this class of problems, it seems reasonable to expect that the stress and strain distribution in the material flowing plastically can be obtained without following through the history of the development of the initial elastic stress distribution as the load is increased to its final value. This is found to be possible in many cases. This class of problems does not, of course, include the determination of the change in the elastic stress distribution when the elastic limit is first exceeded, for the strains throughout are then of the order of elastic strains only.

The stress-strain relationship to be considered therefore comprises zero strain until the yield stress is reached, followed by flow at constant stress. The generalization of this for flow in plane strain, using Tresca's yield limit of flow at constant maximum shear stress, was first given by Saint-Venant [2]. Using rectangular Cartesian coordinates with the flow in plane strain parallel to the xy plane, the Tresca[2] yield condition of constant maximum shear stress k becomes

$$(1) \qquad (\sigma_x - \sigma_y)^2 + 4\tau^2 = 4k^2,$$

where τ is written for the more usual τ_{xy}.

The relationship between stress and flow, or strain rate, is determined by the incompressibility condition, since volume changes are of the order of elastic strains and therefore negligible, and also by the coincidence of the principal directions of stress and strain rate. In terms of the stress and strain rate components, these conditions become

$$(2) \qquad \dot{\epsilon}_x + \dot{\epsilon}_y = 0,$$

$$(3) \qquad \frac{\dot{\epsilon}_x - \dot{\epsilon}_y}{\sigma_x - \sigma_y} = \frac{\dot{\epsilon}_{xy}}{\tau} = \lambda \geqq 0,$$

where λ is a parameter which represents the rate of flow at each point. The strain rate components are defined in the usual way

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[2] The Mises quadratic flow limit [3], which for all combined stress systems agrees with the Tresca limit to within 15 per cent, reduces to the form (1) for plane strain, when strains greater than two or three times the elastic yield-point strain have occurred, and σ_z has settled down to $(\sigma_x + \sigma_y)/2$. See, for example, the elastic-plastic solution for a thick tube in plane strain [4]. However, the use of the Mises criterion calls for greater care in discussing stress conditions at the plastic rigid boundary, although the final solution does not differ significantly from that based on Tresca's flow limit.

$$\dot{\epsilon}_x = \frac{\partial v_x}{\partial x}, \qquad \dot{\epsilon}_y = \frac{\partial v_y}{\partial y}, \quad \text{and} \quad 2\dot{\epsilon}_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x},$$

where v_x and v_y are the velocity components in the x and y directions, respectively. If plastic flow takes place, λ is positive, since shear strain increments occur in the direction of the maximum shear stress. If the material is in a condition of incipient flow, but does not flow, λ is zero.

Stress and strain distributions in quasi-static flow with negligible inertia forces are obtained by combining the plasticity conditions, equations (1) to (3), with the equations of equilibrium, which we shall consider without body forces.

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} &= 0, \\ \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0. \end{aligned} \tag{4}$$

As Hencky [5] pointed out, (1) and (4) form a set of three equations in the three stress components, and so they can be considered as a statically determinate system. However, this concept must be used with reserve, for its validity requires the boundary conditions to be given entirely in terms of stresses. It will be emphasized in Sec. 5 of this paper, that in general only artificial problems can be formulated in this way, actual plastic-flow problems normally being set with mixed boundary conditions. Following Geiringer [6], we shall term the solution for the stress and velocity distributions the complete solution, to be distinguished from a stress solution which satisfies stress conditions only.

The general problem of determining stress distributions satisfying equations (1) and (4) was first treated by Hencky [5]. He showed that considerable simplification results if the Cartesian coordinate system is replaced by curvilinear coordinates α, β, based on the orthogonal system of trajectories of maximum shear stress, which will be referred to as slip lines. Tresca's flow limit then determines the shear stress component $\tau_{\alpha\beta}$ to be constant, equal to k, throughout the field, and the strain rate conditions (2) and (3) reduce to[3]

$$\dot{\epsilon}_\alpha = \dot{\epsilon}_\beta = 0, \tag{5}$$

$$\dot{\epsilon}_{\alpha\beta} \geqq 0. \tag{6}$$

In this formulation, the coordinate lines are the characteristics of both the stress and velocity equations.

Hencky [5] showed[3] that slip-line fields satisfying the stress equations satisfy simple geometrical relationships, which enable such fields to be built up geometrically. Moreover, he showed that the stress distribution can be obtained from the slip-line field by using simple relationships based on the geometry of the system.[3] Thus, a given slip-line field geometry is equivalent to a known stress distribution, and we shall accordingly speak mainly of slip-line fields in discussing stress solutions.

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[3] A brief summary of this development is given in [1], Sec. 3 and the appendix.

For the examples discussed in this paper, we shall need only two simple slip-line fields: orthogonal systems of straight lines which correspond to uniform stress distributions, and circular arcs with corresponding radii. The existence of both these systems was pointed out by Hencky.

2. Stress solutions for a wedge with pressure on one face. In choosing examples by means of which to discuss the physical nature of stress discontinuities, we need consider only the simple problem of straight-sided wedges with uniform pressure on one face, the other face being free of external tractions. The form of the stress solution for wedge angles $2\beta_0 \geqq \pi/2$ was first given by Prandtl [7], and the corresponding slip-line field is shown in Fig. 1*a*. *ABC* is a region of uniform stress, zero shear on *AB* requiring the slip lines to emerge at $\pi/4$ from the boundary *AB*. Similarly the slip-line field in *BDE* has the same form. These two regions are connected by a fan of radii and circular arcs with center *B*. Assuming a continuous stress distribution, this configuration is uniquely determined by the equations governing slip-line fields and the given boundary tractions. The bounding of the region of large strain by the slip-line *ACDE* is determined from the velocity solution, but such a limitation would be expected from the stress solution, since the pressure only extends down the wedge face to the point *A* in Fig. 1*a*. This slip-line field determines that the pressure required to produce large plastic flow is given by

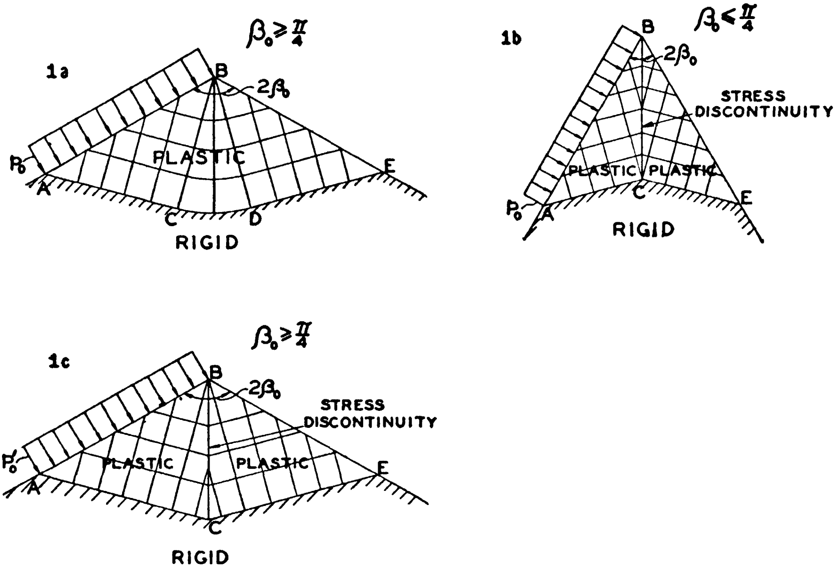

FIG. 1. Slip-line fields for uniform pressure on one face of a wedge.

$$p_0 = 2k\left[1 + \left(2\beta_0 - \frac{\pi}{2}\right)\right]. \tag{7}$$

For wedge angles $2\beta_0 < \pi/2$, the two constant-state regions *ABC* and *BDE* would overlap, and a stress solution of this type cannot be found. Prager [8], however, showed that by permitting between regions of plastic flow a stress discontinuity which satisfies equilibrium conditions,[4] the slip-line field shown in Fig. 1*b* is obtained, which satisfies all the stress conditions of the problem. As shown by Prager [8], the relationships governing the uniform stress regions *ABC* and *BCE*, combined with the equilibrium conditions across the discontinuity *BC*, determine the pressure p_0' required to produce large plastic flow to be

$$p_0' = 2k\left[1 + \sin\left(2\beta_0 - \frac{\pi}{2}\right)\right]. \tag{8}$$

This solution provides a continuation of the continuous solution for wedge angles $2\beta_0$ in the range $0 < 2\beta_0 < \pi/2$, which gives the same solution as the continuous field for $2\beta_0 = \pi/2$. However, this type of solution also satisfies all stress conditions for $\pi/2 < 2\beta_0 < \pi$, providing an alternative stress field, shown in Fig. 1*c*, to that given in Fig. 1*a*. It will be noted that, in this range, p_0' given by equation (8) is less than p_0 given by equation (7), which suggests that, in practice, flow would occur according to the configuration of Fig. 1*c*. However, it will be shown in Sec. 4 that there is no complete solution corresponding to the configuration of Fig. 1*c*, and so it is not an acceptable stress solution for an actual plastic-flow problem.

The difficulty demonstrated by this simple example of alternative stress solutions, when stress discontinuities are considered, is not restricted to this particular problem but represents a general difficulty associated with the consideration of stress discontinuities in plane plastic flow.

3. The conditions at a stress discontinuity. Let us consider the conditions at a stress discontinuity, to see whether additional requirements should be considered which will impose restrictions on the solutions discussed in the previous section.

The stress components in the plastic fields on either side of a stress discontinuity are shown in Fig. 2. Consideration of the equilibrium of a long narrow element containing the discontinuity shows that the normal stress N, and the shear stress T, must be continuous. Equilibrium, however, places no restrictions on the change of N' across the discontinuity, but the plastic-flow condition (1) must be satisfied by the stress system on each side, giving

$$(N - N')^2 + 4T^2 = 4k^2 \tag{9}$$

for both values of N', N_1', and N_2'. Since we are considering the case where a discontinuity of stress exists, N' cannot be continuous, and N_1' and N_2' are therefore given by

$$N' = N \pm 2\sqrt{k^2 - T^2}. \tag{10}$$

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[4] These conditions will be discussed in the next section.

It will be observed that N lies midway between N_1' and N_2', so that permissible elastic or plastic states of stress require $N_1' \leqq N' \leqq N_2'$, for otherwise, with the fixed values of N and T, the maximum permissible yield stress k would be exceeded. Thus, in considering conditions at the discontinuity in detail, we can assume that N' changes from N_1' to N_2', by taking on values N_D' such that $N_1' > N_D' > N_2'$. Equality signs would merely represent extension of the two plastic regions. These inequalities determine

$$(N - N_D')^2 + 4T^2 < 4k^2, \tag{11}$$

since $|N - N_D'| < |N - N_1'| = |N - N_2'|$.

This inequality shows that elastic conditions prevail within the discontinuity, which exists as an elastic membrane between the two plastic regions. When we are referring to the trace of this membrane on a particular plane of flow, the term filament will be used. The reason for considering the existence of a stress distribution in the discontinuity, rather than assuming a strict discontinuity in the mathematical sense, arises from the need to examine the physical existence of such a concept. This requires the Saint-Venant rigid-plastic type of theory to be extended to include the consideration of the influence of elastic strains. The elastic strains associated with a strict stress discontinuity of this nature would violate continuity conditions for the body across the plane of the stress discontinuity.

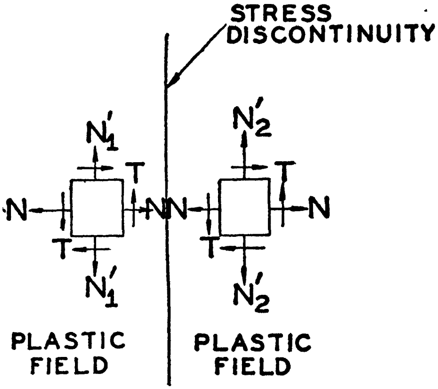

Fig. 2. Stress components at a discontinuity of stress.

The basis for the idealized Saint-Venant equations of plastic flow with which we are dealing is that elastic material can be considered rigid, for we are concerned only with strains large compared with elastic strains, and in general, strains in material still elastic induce strains of the same order only in adjacent plastic material. The case of an elastic filament of infinitesimal width is an exception to this rule. In this case, bending with large changes of curvature is permissible without violating elastic conditions in the fiber, for the strains are of the order (filament thickness/radius of curvature), which will remain small for finite curvature. Thus, in this particular case, we must take into consideration the strains induced in the adjacent plastic material by strains of elastic order in the filament. Average longitudinal strain in the filament, as contrasted with bending strains, can induce strain of the same order only in the adjacent material, and so it can be neglected in our theory.

A stress discontinuity is therefore to be considered as an inextensible membrane embedded between two plastic regions.[5] In the next section, we shall consider what limitations this concept introduces when velocity solutions are considered in combination with stress solutions.

4. Complete solution for a wedge with pressure on one face. Let us first consider the velocity solution for the case of an acute-angled wedge, corresponding to the slip-line field shown in Fig. 1*b*. This field is repeated in Fig. 3, the α, β axes based on the slip-line field also being shown. For this particular example, the α, β system of coordinates is orthogonal Cartesian, which considerably simplifies the equations governing the velocity field. The velocity boundary condition consists of a given distribution of normal velocity on the surface *AB* where the pressure is acting. The boundary *AB* is given by

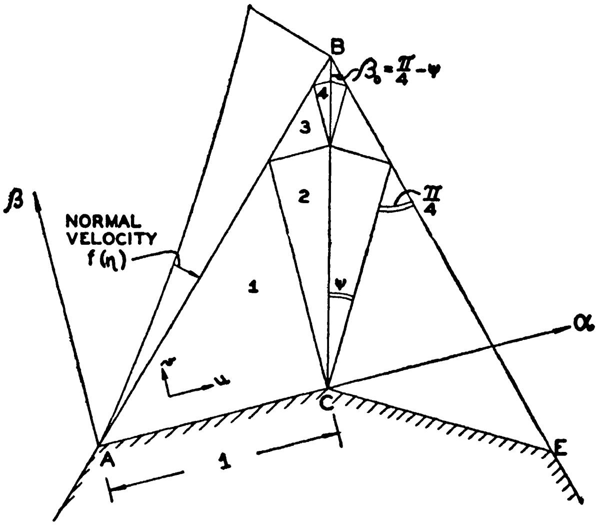

Fig. 3. Slip-line field for uniform pressure on one face of an acute-angled wedge.

$\alpha = \beta = \eta$, and the given normal velocity by $f(\eta)$. With this boundary condition and that determined by the plastic-rigid boundary *ACE*, we require to determine the distribution of velocity throughout the plastic regions. We use the velocity components in the directions α and β, increasing u and v, respectively. As discussed in the previous section, we have the additional condition that the discontinuity *BC* behaves as an inextensible membrane. This, in conjunction with the rigid boundary at *C*, requires the velocity component in the direction *CB* to be zero for points of both plastic fields on the line *BC*.

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[5] One of the conditions on velocities implied by this limitation, that the velocity should be continuous at a stress discontinuity, has been derived independently by R. Hill. See his discussion of a paper by Alice Winzer and G. F. Carrier, Journal of Applied Mechanics vol. 16 (1949) pp. 104, 105.

For this orthogonal Cartesian system of α, β axes, equations (5) can be written

$$\frac{\partial u}{\partial \alpha} = \frac{\partial v}{\partial \beta} = 0 \tag{12}$$

with the solutions

$$u = u(\beta), \qquad v = v(\alpha). \tag{13}$$

On the basis of equations (13), we can build up the velocity solution for the region ABC. On AC, which is taken to be of unit length for convenience, the fixed rigid region below AC requires v to be zero, and since $v = v(\alpha)$, v is zero throughout the triangular region 1 in Fig. 3. We shall write this as

$$v_1 = 0. \tag{14}$$

Since the α, β directions are inclined at $\pi/4$ with AB, the velocity boundary condition on AB is expressed analytically in the form

$$\sqrt{2}\, f(\eta) = u(\eta) - v(\eta). \tag{15}$$

In building up the velocity solution, equation (15) is usually required for determining $u(\beta)$ in a region in which v is already known; for example, the region 1 which we are now considering. Equation (15) is, therefore, more conveniently written in the form

$$u(\beta) = \sqrt{2}\, f(\beta) + v(\beta). \tag{16}$$

Considering the part of the boundary AB in region 1, equation (16) determines $u(\beta)$ along this boundary, which then applies throughout regions 1 and 2 in Fig. 3, since they cover the same range of β. Thus we have

$$u_1 = u_2 = \sqrt{2}\, f(\beta). \tag{17}$$

The condition on the discontinuity surface BC that the velocity component in the direction CB should be zero, takes the form

$$u \sin \psi + v \cos \psi = 0, \tag{18}$$

where ψ is the angle between the direction CB and the direction β increasing. In the development of the solution, this equation is required to determine $v(\alpha)$ when u is known; and using the equation of the line BC,

$$\beta = \frac{\alpha - 1}{\tan \psi},$$

equation (18) can be written in the form

$$v(\alpha) = -tu\left(\frac{\alpha - 1}{t}\right), \tag{19}$$

where t is written for $\tan \psi$. Applying this equation to the part of BC in region 2, we have

$$v_2 = v_3 = -t\sqrt{2}\,f\left(\frac{\alpha - 1}{t}\right), \tag{20}$$

since this expression for v extends over the range of α including regions 2 and 3 of Fig. 3.

Continuing this procedure, making alternate use of equations (16) and (19), the solutions for both u and v can be obtained throughout the region ABC in the form of finite series. As the apex B is approached, the number of terms increases without limit. The limiting expressions have the form

$$\begin{aligned} u &= \sqrt{2}f(\beta) - t\left[\sqrt{2}f\left(\frac{\beta - 1}{t}\right) - t\left\{\sqrt{2}f\,\frac{(\beta - 1)/t - 1}{t} - \cdots -\right\}\right], \\ v &= -t\left[\sqrt{2}f\left(\frac{\alpha - 1}{t}\right) - t\left\{\sqrt{2}f\,\frac{(\alpha - 1)/t - 1}{t} - \cdots - \cdots -\right\}\right]. \end{aligned} \tag{21}$$

The successive arguments of f, in the form of continued fractions, represent the coordinates of points on AB after successive reflection at BC along lines parallel to the coordinate axes.

It is important to check that this solution approaches a limit at the vertex B, for in obtaining solutions to hyperbolic equations in closed domains, it is in completing the solution in such a corner that difficulties might arise. Assuming $f(\eta)$ to be a continuous function, which we will show later to be a necessary condition, we can write f_B, the value of f at B, for the values of f in the sequences (21) for an arbitrary number of terms. This is permissible since successive reflections from a point infinitesimally close to the vertex remain infinitesimally close to the vertex. The sequences then become

$$\begin{aligned} u &= \sqrt{2}f_B(1 - t + t^2 - t^3 - \cdots) = \frac{\sqrt{2}f_B}{1 + t}, \\ v &= -\sqrt{2}f_B(t - t^2 + t^2 - \cdots) = -\frac{\sqrt{2}f_B t}{1 + t}. \end{aligned} \tag{22}$$

We can make use of the trigonometric identity

$$\frac{(1 + t)\cos\psi}{\sqrt{2}} = \cos\left(\frac{\pi}{4} - \psi\right) = \cos\beta_0$$

to obtain the limit of the velocity in ABC at the vertex B. It has a zero component parallel to BC, and $f_B/\cos\beta_0$ normal to BC. The velocity in the plastic field thus approaches the velocity of the elastic filament BC at B, which is determined uniquely by the velocity condition on BC and the given velocity component of B normal to AB.

By symmetry, it is clear that the velocity field obtained for the region ABC can be adapted for the region CDE, leading to the same distribution of normal velocity on EB as that given on AB. Had this not been obvious by symmetry, the solution in CDE would need to have been built out from the known velocity at the vertex B.

Before this solution can be accepted as that for a physical problem of plastic flow, we must check what restrictions are placed on $f(\eta)$ by the condition $\dot{\epsilon}_{\alpha\beta} \geqq 0$ according to equation (5). One would expect some restriction on possible velocity boundary conditions to be associated with the stress field determined by the stress boundary conditions, for the stress field represents shearing of the wedge to the right and the velocity field must correspond with this.

This question can be most easily treated in terms of $\varphi(l)$, the velocity of the elastic filament at the point distance l from C. Since we are considering bending of the elastic filament, $\varphi(l)$ is a continuous function. As indicated on Fig. 4, the velocities at the point α, β in the region ABC are then given directly in the form

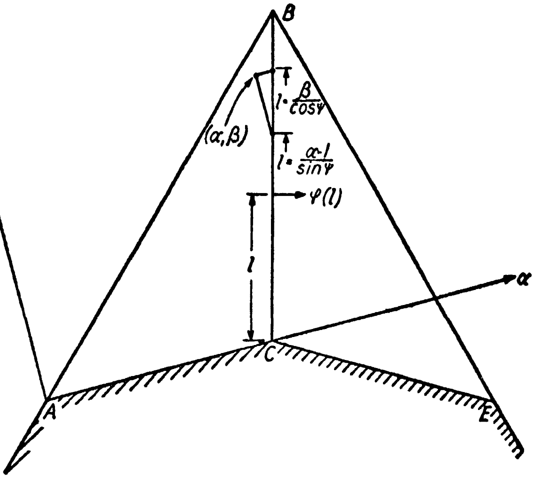

FIG. 4. The velocity field in terms of the velocity of BC.

$$u = \varphi\left(\frac{\beta}{\cos\psi}\right)\cos\psi, \tag{23}$$
$$v = -\varphi\left(\frac{\alpha - 1}{\sin\psi}\right)\sin\psi,$$

$$\dot{\epsilon}_{\alpha\beta} = \frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} = \varphi'\left(\frac{\beta}{\cos\psi}\right) - \varphi'\left(\frac{\alpha - 1}{\sin\psi}\right). \tag{24}$$

Since, in ABC, $\beta/\cos\psi \geqq (\alpha - 1)/\sin\psi$, equation (5), $\dot{\epsilon}_{\alpha\beta} \geqq 0$, requires $\varphi'(l)$ to be a nondecreasing function of l. In the region 1 of Fig. 3, where $v = 0$, the condition $\dot{\epsilon}_{\alpha\beta} \geqq 0$ requires $\varphi'(\beta/\cos\psi) \geqq 0$ over the region of CB adjacent to region 2 of Fig. 3. Combining these conditions with the condition at C, $\varphi(0) = 0$, we have

$$\varphi(0) = 0, \qquad \varphi'(l) \geqq 0. \tag{25}$$

$\varphi'(l)$ is a nondecreasing function.

Making use of equation (15), $f(\eta)$ must therefore be expressible in the form

$$\sqrt{2}\, f(\eta) = \varphi\left(\frac{\eta}{\cos\psi}\right)\cos\psi + \varphi\left(\frac{\eta - 1}{\sin\psi}\right)\sin\psi H(\eta - 1),$$

where $H(\xi)$ is the unit step function, and $\varphi(l)$ satisfies the conditions stated in (25).

The physical interpretation of (25) is that the rate of change of curvature of the elastic filament must be increasing toward the apex in order to maintain positive strain rate throughout the field.

Having obtained the complete solution, it is interesting to point out that a solution of this type has been known for many years, without being interpreted as a special case of the general problem of stress discontinuities in plane plastic flow. The development of the stress distribution across a slab bent by couples can be followed from elastic bending until large plastic strains occur (see [9]), as illustrated in Fig. 5. Figure 5*a* shows the linear elastic stress distribution. As the moment is increased, elements close to the surfaces of the slab flow plastically, and we have the situation shown in (*b*) with an elastic core. As the bending moment increases further toward the maximum permissible value corresponding to the configuration (*c*), the elastic core shrinks to a membrane, and we have the special case of a discontinuity with a saltus of $4k$ in the stress component parallel to it. This particular example represents the largest possible change in stress at a discontinuity.

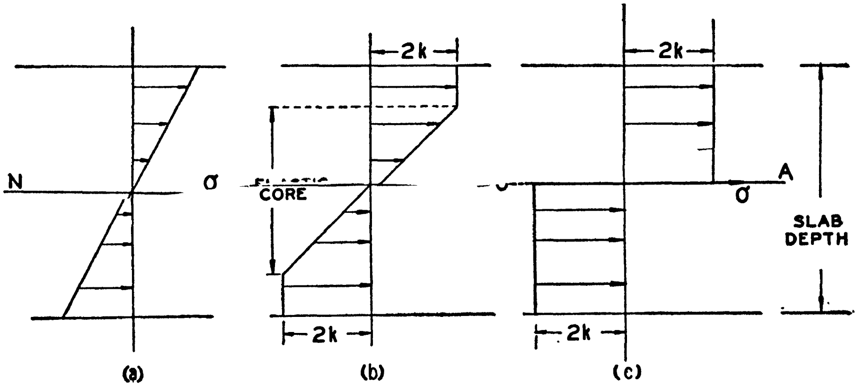

FIG. 5. Development of the stress distribution in a bent slab.

Let us now consider the velocity solution corresponding to Fig. 1*c*. This configuration is repeated in Fig. 6, with the slip-line field coordinate axes α, β included. The form of the solution given in equations (13), and the boundary conditions, equations (16) and (19), apply for this case also. The plastic rigid boundary AC requires $v = 0$, which in view of the second of equations (13), determines v to be zero throughout regions 1 and 2 of Fig. 6. Combining this with the boundary condition on BC, equation (19), determines u to be zero on BC, and so throughout regions 1 and 2, in view of the first of equations (13). This solution, $u = v = 0$, is compatible only with $f(\eta) = 0$ along AB, which does not represent a problem of plastic flow.

If we consider the continuous stress field for the obtuse-angled wedge illustrated in Fig. 1a, a satisfactory velocity solution can be found (see Sec. 4 of [1]) if $f(\eta)$ is a nondecreasing function.

Thus for the range of problems we have considered with wedge angles $2\beta_0$ between zero and π, a single set of complete solutions is obtained. For $0 < 2\beta_0 < \pi/2$, only solutions with stress discontinuities satisfy the conditions, and for these the velocity fields are continuous. For $\pi/2 < 2\beta_0 < \pi$, the stress solution containing a discontinuity cannot be associated with a velocity solution, and so it does not represent a physical problem of plastic flow. In this region, the continuous stress field produces a complete solution, and it is interesting to observe that in this case discontinuities of velocity may exist across characteristics.

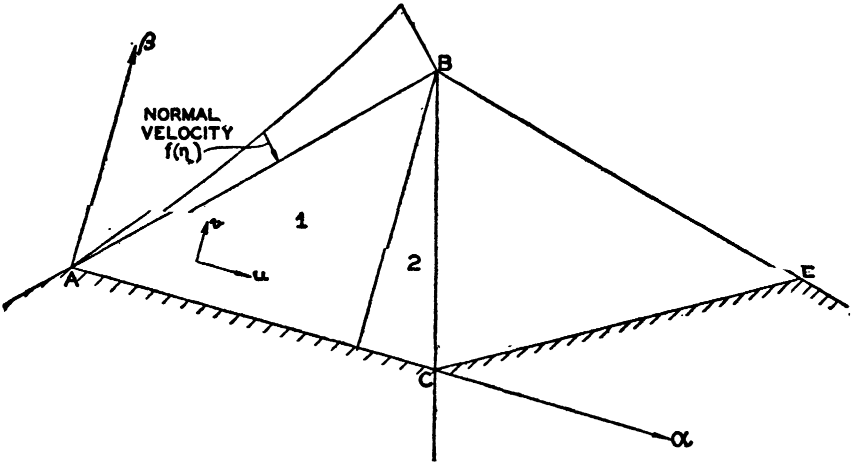

Fig. 6. Slip-line field for uniform pressure on one face of an obtuse-angled wedge.

Thus, for this set of problems, consideration of the complete solution avoids the difficulty of alternative solutions which arise when stress solutions only are considered. The velocity conditions imposed by a stress discontinuity, which were developed in the previous section, are not restricted to this particular example but must be used in obtaining the complete solutions for all cases of stress discontinuities in plane plastic flow.

5. General remarks on plane strain problems. The questions discussed in the previous section illustrate an important feature of problems of plastic flow in plane strain based on the Saint-Venant theory of an ideal plastic material. We saw that a stress solution alone, based on stress boundary values, may not represent a physical problem of plastic flow. A complete solution must be obtained which satisfies both stress and strain rate plasticity conditions [equations (1) to (3)] before it can be accepted as a satisfactory plastic-flow problem, or in fact before we know what concrete physical problem has been solved, for an infinity of velocity solutions may correspond with a single stress field. Thus we must make a distinction between (*a*) solutions of the stress equations and (*b*) solutions of plastic-flow problems.

As was pointed out in Sec. 1 of this paper, Hencky observed that the stress equations (the equilibrium equations with the flow limit) form a statically determinate system if stress boundary conditions are given. Following Hencky, most authors have investigated this type of problem, obtaining stress fields without obtaining corresponding velocity solutions. As will be emphasized later in this section, this method of attack has not been fruitful in treating practical plastic-flow problems, for they are in general set with mixed stress and velocity boundary conditions. Moreover, difficulties have arisen in the solution of stress problems in that limit lines are reached beyond which the solution cannot be continued (see, for example, the work of Christianovich [10]). These have been termed lines of rupture and attributed with a physical significance, but it will be shown in the following discussion that they represent purely mathematical difficulties associated with the manner in which the boundary-value problems have been set.

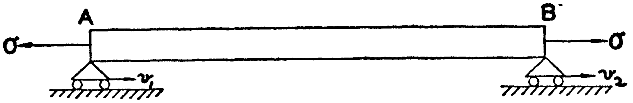

FIG. 7. A rod strained under longitudinal stress.

To illustrate this point, let us consider the simple system shown in Fig. 7, consisting of a rod AB of ideally plastic material subjected to a longitudinal stress σ, the ends moving with velocities v_1 and v_2, respectively. Let the magnitude of the flow stress in tension and compression be Y. Then we have to consider three situations defined in terms of the stress σ:

(i) $|\sigma| < Y$. The bar is elastic.
(ii) $|\sigma| = Y$. Plastic flow can occur in tension or compression depending on the sign of σ.
(iii) $|\sigma| > Y$. No equilibrium or quasi-static configuration exists for the ideally plastic material.

Thus for only one particular value of $|\sigma|$ does a plastic-flow problem exist, and a plastic-flow problem can only be set in terms of stresses by applying this value of $|\sigma|$ as the boundary condition.

This situation is avoided if the end conditions are given in terms of velocities, for, neglecting elastic strains, whenever $v_2 > v_1$ we have plastic flow in tension, and whenever $v_1 > v_2$, plastic flow in compression. Thus in this case, no precision is required to set a plastic-flow problem in terms of velocities.

A similar state of affairs exists in setting problems of plane strain, but in this case the precise conditions on boundary stresses required to set a plastic-flow problem are not known; any particular problem set in terms of stresses may be elastic, or no quasi-static solution may exist. The normal method of procedure, following the concept of static determinacy, has been to assume the existence of a plastic stress solution, to use the given surface tractions in combination with the flow condition to determine completely the stress tensor along the boundary, and then to build up the solution along the characteristics from the boundary. In general, a limit is reached beyond which the solution cannot be extended, the difficulties arising in a more obvious form when the solution is sought in the interior of a closed boundary. However, this cannot present a physical difficulty in the analysis of plastic-flow problems, for the material must be in either the elastic or the plastic state, and the analysis of a physical problem must be possible in terms of regions satisfying either the elastic or the plastic conditions.

When the manner in which physical problems are set is studied, this difficulty is seen not to arise, for such problems are always set in part by velocity conditions. For example, in the metal-forming problems of rolling and extrusion, plastic flow is generated by velocity conditions imposed by the rigid rolls or the moving die, respectively, and in the case of a rigid wedge indenting into a plastic body, velocity conditions at the surface of the wedge generate the plastic flow. For such cases, the complete problem must be considered as an entity, which comprises a system of two first-order partial differential equations in stresses, and two in velocities, the two pairs being hyperbolic, both having the slip lines for characteristics. It is the solution of these equations, combined with the determination of the plastic-rigid boundary, which typifies the types of solution of plane plastic-flow problems which will be of value in the theoretical investigation of technological problems. This point of view is emphasized here, for it is hoped that the appreciation by mathematicians that the theoretical analysis of plane plastic flow requires solutions of this set of four equations, rather than merely the stress equations which have been thoroughly treated, may lead to the development of methods of solution. For example, it would be of great interest to determine mathematical conditions on the types of boundary-value problems which lead to satisfactory solutions of these hyperbolic equations in closed domains, instead of having to rely on physical intuition for the form in which such boundary-value problems are properly set. This general question of complete solutions has been discussed, and some particular examples have been presented in recent years [1, 11, 12, 13, 14, and 15].

6. Approximate solutions. In previous sections, we have been concerned with stress discontinuities in plane plastic flow which we showed to exist in the physical form of elastic membranes embedded between domains of plastic flow. Such solutions satisfied all the Saint-Venant relationships governing plastic flow and therefore represent approximate solutions of actual plastic-flow problems to the extent that the idealized laws of Saint-Venant represent the behavior of the actual material. If we satisfy the Saint-Venant relationships only approximately, we can expect solutions which approximate to the type of solution which we have been considering previously. Hodge [16] has used stress discontinuities in this way to obtain simple approximations to plastic deformation problems in plane strain.

The discussion in Sec. 3 of the conditions at a stress discontinuity showed that its deformation which influenced the adjacent plastic material took the form of bending, so that no discontinuity of tangential velocity across a stress discontinuity is permissible. Study of the velocity conditions, equations (2) and (3), shows that a tangential discontinuity of velocity can occur only across a slip line, which is a characteristic of the velocity equations. It would seem, therefore, to provide a reasonable approximation if we permit discontinuities of tangential velocity across lines of stress discontinuity, if the directions of those lines are close to either of the characteristic directions in the plastic fields. Inasmuch as the shear stress across a slip line is k, the transformation of stress components at a point shows that the direction of a stress discontinuity and a slip-line direction lie close together when the shear stress T, shown in Fig. 2, is only slightly less than k. In fact, Prager has shown [8] that a stress discontinuity makes equal angles φ with the slip-line directions in the adjacent plastic fields, where $\cos 2\varphi = T/k$. Equation (10) shows that when T is only slightly less than k, $N_1' - N_2'$ is small, and it is convenient to speak of this difference as the strength of the discontinuity, and thus to term weak discontinuities those associated with small φ. Hodge [16] has shown that if stress fields are built up consisting of regions of uniform stress separated by stress discontinuities, velocity solutions take on very simple forms if velocity discontinuities are permitted across the stress discontinuities. If stress solutions involving only weak stress discontinuities are used, the resulting velocity solution will be expected, with the stress field, to form a useful approximation to the complete solution. For example, the complete solution for the obtuse-angled wedge problem of Sec. 2 can be approximated with one stress discontinuity on the basis of the stress field shown in Fig. 1c. If, however, the fan of the continuous stress solution is replaced by a pencil of stress discontinuities, they can be chosen to lie in closer proximity to the adjacent slip-line directions and so provide a closer approximation to the complete solution.

It has been found that, in general, such approximate solutions require slightly smaller external tractions to produce plastic flow, as is shown, for example, by comparison of equations (7) and (8) for the obtuse-angled wedge. This reduction is to be associated with the relaxation of constraints in the system, in the form of permitting tangential discontinuities of velocity across lines of stress discontinuity, where the shear stress is less than k, instead of only across slip lines where the maximum shear stress k acts. As would be expected, when weak stress discontinuities are used, the corresponding reduction in external tractions is small, indicating a close approximation to the complete solution.

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INDEX

A

Allen, D. N. de G., 199
Anisotropic elasticity, 1–11
Approximate methods, 1
Asymptotic integration, 53

B

Bahšiyan, F. A., 77
Baker, T. J., 80
Bars in tension, 199
Barten, H. J., 74
Bateman, H., 186
Bauschinger effect, 67
Beam, 142
Beams, concentrated loading, 13
Bending of plates, 125
Bers, L., 186
Biezeno, C. B., 27
Biharmonic equation, 129
Biharmonic function, 135
Bijlaard, P. P., 86
Biot, M. A., 67, 85
Bisshopp, K., 74
Blaschke, W., 161
Blumenthal, O., 53
Bolle, L., 117
Boller, K., 85
Boussinesq, J., 108
Brandstatter, J., 8
Bridgeman, P. W., 73
Brillouin, L., 67
Brillouin, M., 67
Bromberg, E., 27
Bromwich integral, 172

C

Carrier, G. F., 125–128, 219
Carrying capacity, 196
Chien, W. Z., 132
Christianovich, S., 225
Cisotti, U., 67
Coburn, N., 201–211
Colonnetti, G., 193
Conduction problems, 165
Conformal mapping, 19
Conway, H. D., 74
Cosserat, E., 67
Cosserat, F., 67
Courant, R., 144
Cox, H. L., 86

D

Davids, N., 26
de Bruyne, N. A., 85
Deuker, E.-A., 67
Diaz, J. B., 161
Difference equations, 204
Dirichlet's integral, 141
Dirichlet's principle, 145
Dirichlet's problem, 141, 145
Discontinuities of stress, 213
Dokos, S., 229
Donnell, L. H., 27, 229
Drucker, D. C., 74
Dynamic loads, 107

E

Eigenfunctions, 125
Eigenvalue problem, 129
Elam, C. F., 85
Elastic failure, 78
Elastic foundations, 107
Elastic-plastic straining, 199–200
Elastic-plastic structures, 187–197
Elastic subgrade, 114
Electric fields, 164
Elliptic integrals, 182
Ericksen, W. S., 85
Eulerian method, 67

F

Facings of sandwich column, 85
Feinberg, S. M., 196
Finite deflections, 27
Finite elastic deformation, 67–84
Finite twist, 75
Flexure, finite pure, 73
Föppl, L., 52
Fourier transforms, 114
Friedrichs, K. O., 42, 117–124, 153

G

Gelbart, A., 186
Gleysal, A., 67
Goodier, J. N., 86
Gorgidze, A. J., 76, 81
Gough, C. S., 85
Graphical method, 201
Green, A. E., 5
Greenberg, H. J., 193

Green's formula, 144
Guest, J. J., 79

H

Haar, A., 195
Hamilton, O. H., 141
Hankel transform, 112
Happel, H., 108
Harmonic functions, 164, 165, 185
Hay, G. E., 13–26
Hencky, H., 215
Hertz, H., 108
Higgins, T. J., 143
Hildebrand, F. B., 53–66
Hill, R., 219, 228
Hodge, P. G., 220, 227
Hoff, N. J., 86
Holgate, S., 5
Holl, D. L., 107–116
Hopkins, H. G., 85
Hydrostatic pressure, uniform, 73

I

Influence functions, 108

J

Jaramillo, T. J., 229
Jeffery, G. B., 26
Jen, S., 131

K

Kappus, R., 67
Kelvin, Lord, 117
Kelvin's theorem, 147
Kirchhoff, G., 117
Kirchhoff boundary condition, 117–124

L

Lagrangian method, 67
Lechnitzky, S. G., 1, 5
Lee, E. H., 213–228
Leggett, D. M. A., 85
Levy, M., 117
Levy, S., 138
Limit design, 195
Lin, C. C., 45
Load, moving concentrated, 111
  pulsating, 115
Loading, proportional, 195
Love, A. E. H., 26, 60, 88, 117, 142, 186
Luxenberg, H., 8

M

McLachlan, N. W., 137
March, H. W., 85–106
Massau method, 201
Mautner, S. E., 86
Meissner, E., 52, 53
Membrane solution, 61
Membrane stresses, 132
Membrane theory, nonlinear, 35
Michlin, S. G., 5
Miesse, C. C., 229
Milne-Thompson, L. M., 81
Mindlin, R. D., 26
Mises-Hencky criterion, 78, 199
Morkovin, V., 11
Moufang, R., 68
Murnaghan, F. D., 67
Muschelisvili, N. I., 1, 5, 14, 16, 25

N

Nádai, A., 128, 228
Nash, W., 139
Navier edge conditions, 113
Neal, B. G., 197
Nelson, C. W., 26
Neuber, H., 86
Neumann's problem, 141, 146
Newing, S. T., 81
Nonlinear membrane theory, 35
Nonlinear theory, 27
Norris, C. B., 85

P

Panov, D., 67
Perry, C. L., 131–139
Perturbation methods, 5
Plastic enclaves, 199
Plastic flow, 213–228
Plasticity, plane, 201–211
Plate, anisotropic infinite, 109
  circular, 113
  flat, 33
  prestressing, 129
  rectangular, 110
Plate equations, 107, 132, 134
Plates, circular, 131
  edge effect for, 117–124
  elliptic, 131–139
  thin, 107–117, 125–128
Pollaczek-Geiringer, H., 215
Polya, G., 141, 154
Poritsky, 163–186

Potential, 181
Prager, W., 153, 187–197, 200, 206, 211, 217
Prandtl, 216
Prandtl-Reuss flow theory, 199
Principal directions, 202
Proportional loading, 195
Pulsating load, 115

R

Reiner, M., 69
Reissner, E., 27–52, 65, 85, 117
Reissner, H., 46, 53
Relaxational methods, 199–200
Richter, H., 68
Ring, current through, 181
  torque over, 179
Ripperger, E. A., 26
Rivlin, R. S., 67
Riz, P. M., 67
Ruchadze, A. K., 76, 81

S

Safety factor, 196
Saint-Venant, B. de, 154, 214
Sandwich columns, 85
Schleicher, F., 108
Schwarz, H. A., 159
Schwarz's inequality, 144
Schwarz-Christoffel tensor, 68
Seth, B. R., 67–84
Shafts, 163, 172
Shaw, F. S., 125–128, 200
Shells, cylindrical, 27, 33, 34, 37
  of revolution, 27, 53
  spherical, 27, 39, 41, 59
  theory of, 53
  thin, 27
Shepherd, W. M., 77, 81
Sherman, D. I., 5
Signorini, A., 67
Smith, C. B., 85
Sokolnikoff, I. S., 1–11, 16, 142
Sokolovsky, W. W., 201
Southwell, R. V., 199–200
Sphere, hollow, 77
Spherical harmonics, 168
Sternberg, E., 69
Steuermann, E., 53
Stevenson, A. C., 5, 26
Stiffness, 143
Stoker, J. J., 27, 129
Stone, M., 161
Strain energy tensor, 68
Stress, discontinuities of, 213
Stress fields, 163–186
Stress function, 1, 14, 88, 133, 163, 164
  Prandtl's, 155
Stress space, 189
Stress-strain relations, 69
Sturm-Liouville systems, 136
Subspace, of compatibility, 190
  of elastic states, 190
  of residual states, 191
Swainger, K. H., 69, 74
Symonds, P. S., 187–197
Synge, J. L., 132, 153
Szego, G., 154

T

Tait, P. G., 117
Taylor, G. I., 5
Timoshenko, S., 15, 116, 128
Timpe, A., 26, 137, 186
Tolotti, C., 69
Tonolo, A., 67
Torque, 171
Torsion, 163
  elastic-plastic, 200
Torsional rigidity, 141–161
Trefftz, E., 145
Trefftz's method, 145
Treloar, L. R. G., 72
Trusses, 187
Tsien, H. S., 27
Tupper, S. J., 228

V

Vaisey, G., 199
Van der Neut, A., 86
Voigt, W., 69
von Kármán, T., 27, 195, 211
von Mises, R., 214

W

Wan, C. C., 86, 229
Warping function, 142
Weinstein, 127, 141–161, 182
Westergaard, H., 108
Williams, D., 85
Wilson, W., 210
Winkler, E., 108
Winzer, A., 219
Wrinkling, 85
  antisymmetrical, 93
  symmetrical, 96

Z

Zvolinsky, N. V., 67